Cal Poly Department of Mathematics

Puzzle of the Week

April 25 - 30, 2009

Let B denote a box in 3-space. Suppose B can be partitioned into a finite number of smaller boxes such that each of the smaller boxes has at least one dimension of integer length. Show that B must have at least one dimension of integer length.

[Note + Hint: There are perhaps numerous ways to approach this tricky puzzle, but let me suggest one which involves a little multi-variable calculus: consider integrating a suitable function over B]

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: The two-dimensional version of this problem was the subject of an American Mathematical Monthly article by Stan Wagon entitled "Fourteen proofs of a result about tiling a rectangle". As the title suggests there are over a dozen ways to approach this problem!

A friend of mine suggested the three-dimension version presented here, with the given hint. The idea is this: if

$$f(x, y, z) = \sin(2\pi x)\sin(2\pi y)\sin(2\pi z)$$

and B' is any box in 3-space with edges parallel to the axes then $\int_{B'} f\,dxdydz = 0$ if any side length of B' is an integer. To see this let the interval $[a, b]$ have integral length (i.e. $b - a$ is an integer). Then:

$$\int_a^b \sin(2\pi x)\,dx = \frac{-1}{2\pi}\left(\cos(2\pi b) - \cos(2\pi a)\right) = 0$$

by the periodicity of $\cos(2\pi x)$. Now if $B' = [a,b] \times [c,d] \times [e,f]$ using iterated integration we find

$$\iiint_{B'} f(x,y,z)\,dxdydz = \int_a^b \sin(2\pi x)\,dx \int_c^d \sin(2\pi y)\,dy \int_e^f \sin(2\pi z)\,dz = 0$$

if any of $[a,b], [c,d], [e,f]$ are integer length.

Now let $B = \cup_i B_i$ be the given (finite) partition and place B parallel with the axes, one corner at the origin, and in the first orthant. On the one hand

$$\iiint_B f\,dxdydz = \sum_i \iiint_{B_i} f\,dxdydz = 0$$

since each B_i is assumed to have at least one side of integer length.

On the other hand, if B has side lengths of a, b, c then

$$\begin{aligned}\iiint_B f\,dxdydz &= \int_0^a \sin(2\pi x)\,dx \int_0^b \sin(2\pi y)\,dy \int_0^c \sin(2\pi z)\,dz \\ &= \frac{-1}{(2\pi)^3}\left(\cos(2\pi a) - 1\right)\left(\cos(2\pi b) - 1\right)\left(\cos(2\pi c) - 1\right).\end{aligned}$$

So one of the above terms must be zero. Say $\cos(2\pi a) = 1$. Then $2\pi a = 2\pi n$ for some integer n, and the solution follows.