

Cal Poly Department of Mathematics

Puzzle of the Week

April 17 - 23, 2009

Following the revolution, the queen of Greediopia (population 130, including her Highness) has lost all power including the right to vote, but she does retain the power to suggest redistribution of salaries. Each suggestion is voted upon by citizens, and if there are more votes in favor than against the new salary distributions are put in place. Each resident (including the queen) begins with a salary of one dollar, salaries are always whole numbers, and the sum of all salaries at any time must remain 130.

Assume each resident is perfectly greedy: he or she can be counted upon to vote in favor of a redistribution if his or her salary goes up, vote against it if it goes down, and not to bother voting if it does not change. What is the maximum salary a clever queen can get for herself, and how many votes must she call before she gets it?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: I found this puzzle in "Mathematical puzzles: a connoisseur's collection" by Peter Winkler (who attributes it to Johan Wästlund of Linköping University, inspired very loosely on historical events in Sweden).

The queen can get 127 dollars and needs at least eight votes to do it. The idea is to always offer just over half of the salaried voters a raise; such a vote will always pass. When the number of salaried voters dwindles to a minimum a few paupers can be bribed to out-vote them. To start things off the queen must actually give up her salary:

In the first vote the queen suggests that 65 citizens each get a raise to two dollars each,

while the other 64 and herself are reduced to no salary.

In the second vote the queen suggests that 33 of the 65 voters with salaries receive a raise, while the other 32 of them be reduced to zero dollars. Everyone else's salary is unchanged (though the queen may at this time take some for herself if she like).

In the third vote 17 of the 33 with salaries get raises, everyone else goes to (or remains at) zero.

Repeating this will lead to 2 citizens with salaries after 7 votes. Since they received raises at each vote they make at minimum 8 dollars apiece. Now the queen offers three un-salaried voters one dollar apiece, all other citizens zero dollars, and herself the remaining \$127.

To see that 127 dollars is the most money the queen can get note that it is impossible to end up with only one (or fewer) salaried citizens (since he or she could not out-vote any number of people the previous round). So the queen cannot get 130 or 129 dollars. To end up with 128 there would have to be two citizens with one dollar apiece, but then the previous vote would have had them out-voting a single citizen with the only salary – a situation we know is impossible.

To see that eight votes are necessary notice that if the queen has 127 dollars then prior to the last vote there must have been only two citizens with salaries. But one cannot get to such a situation in under seven votes since each vote can eliminate at most just under half the salaries.