Cal Poly Department of Mathematics

Puzzle of the Week

Apr 10 - 16, 2009

We all know the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. What if we removed all the terms with a 9 in the denominator (in its decimal expansion)... would the new series be convergent or divergent?

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: Let s_k denote the sum of the reciprocals of those numbers with k digits which have no 9 in their decimal expansion. Since any such n is greater than (or equal to) 10^{k-1} (a one followed by $k-1$ zeroes) we see that s_k is bounded above by $1/10^{k-1}$ times the number of such n, which is $8 \cdot 9^{k-1}$ (there are eight choices for the leading digit: 1-8, and nine choices – 0-8 – for each other digits). That is

$$s_k \leq \frac{8 \cdot 9^{k-1}}{10^{k-1}} = 8 \cdot \left(\frac{9}{10}\right)^{k-1}.$$

Now

$$\sum_{n \text{ with no } 9} \frac{1}{n} = \sum_{k=1}^{\infty} s_k \leq \sum_{k=1}^{\infty} 8 \cdot \left(\frac{9}{10}\right)^{k-1} = \frac{8}{1-9/10} = 80.$$

Thus the sum is convergent.

Jonathan Shapiro points out that the same argument works with any string of numbers in place of 9. For example the sum over all those n which do not contain 1234567890314159 anywhere in their decimal expansion will also converge, and in fact can be bounded by a geometric series.

Jim Delany gives a sharper estimate by using $n \geq L \cdot 10^{k-1}$ where L is the leading digit of n. Using this one finds the above sum is actually less than 28.