Cal Poly Department of Mathematics

Puzzle of the Week

Apr 3 - 9, 2009

Recall the parabola $y = x^2$ has focus at $(0, 1/4)$. If we roll the parabola (without slipping) along the x-axis what is the locus of points traced out by the focus?

Solutions should be submitted to Morgan Sherman:

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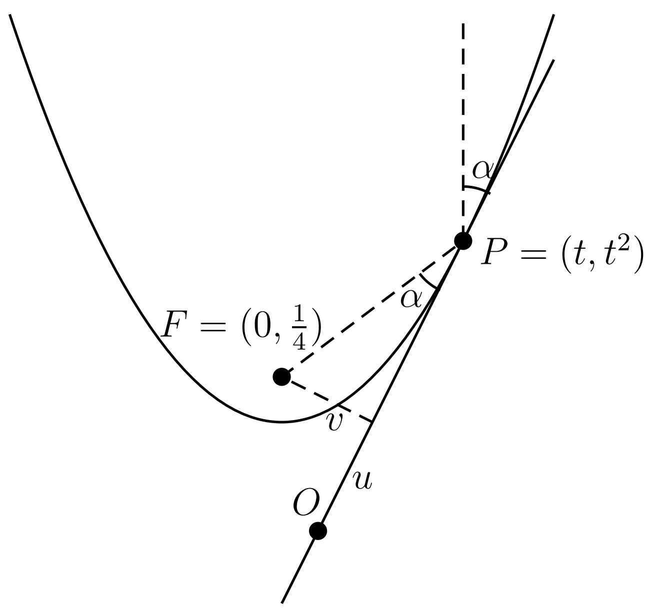
before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: Keep the xy-coordinates fixed in the reference frame of the parabola and suppose the point $P = (t, t^2)$ is in contact with the "ground". From the point of view of the parabola the ground is the tangent line at P. Let O denote the origin and (u, v) the focus in the reference frame of the ground. See the diagram below and note the following two facts:



1. The condition “rolls without slipping” implies $|OP| =$ the arclength along the parabola from $(0,0)$ to (t,t^2). From the formula for arclength from calculus we calculate

$$|OP| = \int_0^t \sqrt{1+4t^2}\,dt = \frac{t}{2}\sqrt{4t^2+1} + \frac{1}{4}\sinh^{-1}(2t).$$

2. The focus of a parabola satisfies the property that any ray of light shining straight down onto the parabola will reflect and pass through the focus. Hence the two labeled angles (α) in the diagram are equal. Thus

$$\cot\alpha = \text{ slope of the tangent line } = 2t.$$

From this we find

$$\sin\alpha = \frac{1}{\sqrt{4t^2+1}}, \qquad \cos\alpha = \frac{2t}{\sqrt{4t^2+1}}.$$

Now we are ready. From the diagram we have:

$$v = |FP|\sin\alpha = \left(t^2 + \frac{1}{4}\right)\frac{1}{\sqrt{4t^2+1}} = \frac{1}{4}\sqrt{4t^2+1}$$

and

$$\begin{aligned} u &= |OP| - |FP|\cos\alpha \\ &= \left[\tfrac{t}{2}\sqrt{4t^2+1} + \tfrac{1}{4}\sinh^{-1}(2t)\right] - \left(t^2 + \tfrac{1}{4}\right)\tfrac{2t}{\sqrt{4t^2+1}} \\ &= \tfrac{1}{4}\sinh^{-1}(2t) \end{aligned}$$

Therefore

$$v = \frac{1}{4}\sqrt{(2t)^2+1} = \frac{1}{4}\sqrt{\sinh^2(4u)+1}$$

or

$$v = \frac{1}{4}\cosh(4u)$$

This curve describes the famous “catenary” (e.g. the shape of a hanging telephone wire).