Recall the parabola \( y = x^2 \) has focus at \((0, 1/4)\). If we roll the parabola (without slipping) along the \( x \)-axis what is the locus of points traced out by the focus?

**Solution:** Keep the \( xy \)-coordinates fixed in the reference frame of the parabola and suppose the point \( P = (t, t^2) \) is in contact with the “ground”. From the point of view of the parabola the ground is the tangent line at \( P \). Let \( O \) denote the origin and \((u, v)\) the focus in the reference frame of the ground. See the diagram below and note the following two facts:
1. The condition “rolls without slipping” implies $|OP|$ = the arclength along the parabola from $(0,0)$ to $(t,t^2)$. From the formula for arclength from calculus we calculate

$$|OP| = \int_0^t \sqrt{1 + 4t^2} \, dt = \frac{t}{2} \sqrt{4t^2 + 1} + \frac{1}{4} \sinh^{-1}(2t).$$

2. The focus of a parabola satisfies the property that any ray of light shining straight down onto the parabola will reflect and pass through the focus. Hence the two labeled angles ($\alpha$) in the diagram are equal. Thus

$$\cot \alpha = \text{slope of the tangent line} = 2t.$$ 

From this we find

$$\sin \alpha = \frac{1}{\sqrt{4t^2 + 1}}, \quad \cos \alpha = \frac{2t}{\sqrt{4t^2 + 1}}.$$ 

Now we are ready. From the diagram we have:

$$v = |FP| \sin \alpha = \left( t^2 + \frac{1}{4} \right) \frac{1}{\sqrt{4t^2 + 1}} = \frac{1}{4} \sqrt{4t^2 + 1}$$

and

$$u = |OP| - |FP| \cos \alpha$$

$$= \left[ \frac{1}{2} \sqrt{4t^2 + 1} + \frac{1}{4} \sinh^{-1}(2t) \right] - \left( t^2 + \frac{1}{4} \right) \frac{2t}{\sqrt{4t^2 + 1}}$$

$$= \frac{1}{4} \sinh^{-1}(2t)$$

Therefore

$$v = \frac{1}{4} \sqrt{(2t)^2 + 1} = \frac{1}{4} \sqrt{\sinh^2(4u) + 1}$$

or

$$v = \frac{1}{4} \cosh(4u)$$

This curve describes the famous “catenary” (e.g. the shape of a hanging telephone wire).