Cal Poly Department of Mathematics

Puzzle of the Week

Feb 20 - 26, 2009

Fix a line segment l in the plane. Among all triangles which have l as longest side one is chosen at random. What is the probability it is obtuse?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

<u>*Solution:*</u> I found this problem in Lewis Carroll's "Pillow Problems" (loaned to me by Joe Borzellino) where the statement simply read "what is the probability a random triangle is obtuse". Unfortunately there are many possible interpretations of a "random triangle" (see the last paragraph) so I included the fixed line segment l to try and corral people into the following solution:

Since l is fixed in the plane a triangle having l as a side is determined by the location of the opposite vertex C. Let the endpoints of l be denoted A and B. There is no loss in generality if we take the length of l to be 2. Then if l is the longest side of the triangle ABC it must be that C lies within the circle of radius 2 centered at A (so that $|AC|$ does not exceed 2) and similarly within the circle of radius 2 centered at B. Call this region R and denote the points of intersection of these two circles by P, Q (see the figure below). Then R has area:

$$\{\textit{area of sectors } PAQ\} + \{\textit{area of sector } PBQ\} - \{\textit{area of triangles } ABP, ABQ\}.$$

Notice that triangle ABP (as well as triangle ABQ) is equilateral with side length 2. In particular it follows that angle PAQ (as well as PBQ) is $\frac{2\pi}{3}$ and that each triangle has area (by Heron's formula, for instance) $\sqrt{3}$. So the region R of all possible locations for the vertex C has area

$$2 \times \left\{\frac{\pi}{3}2^2\right\} - 2 \times \{\sqrt{3}\} \quad = \quad \frac{8\pi}{3} - 2\sqrt{3}$$

Now given a vertex C within the region R the angles at vertices A and B will be $< \frac{\pi}{2}$. So the triangle will be obtuse if and only if the angle θ at vertex C is $> \frac{\pi}{2}$. By the Law of Cosines we find

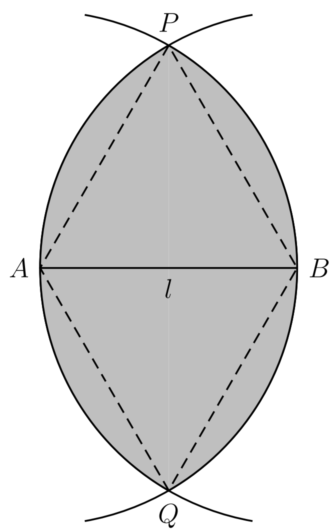
$$\cos\theta = \frac{|AC|^2 + |BC|^2 - |AB|^2}{2|AC||BC|} < 0 \iff |AC|^2 + |BC|^2 < |AB|^2$$

Establish coordinates so that l lies horizontally centered at the origin and let $C = (x, y)$. Then the above inequality becomes:

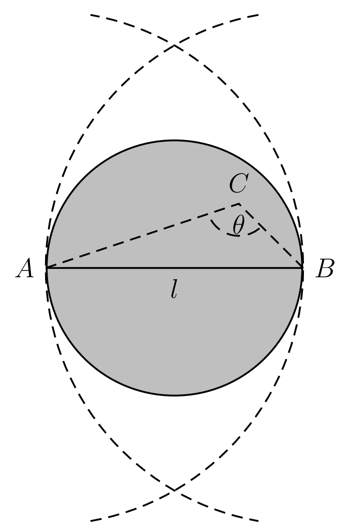
$$\left\{(x+1)^2 + y^2\right\} + \left\{(x-1)^2 + y^2\right\} < 4$$

which reduces to $x^2 + y^2 < 1$. So the triangle is obtuse iff C lies inside the circle with diameter l, which has area π (see the figure below). The probability a random triangle is obtuse is therefore the ratio of this area to the area of the region R, which gives the formula

$$\frac{3\pi}{8\pi - 6\sqrt{3}} \approx 63.94\%.$$



The region R



The region of obtuse triangles

It should be noted that one can still argue about what is meant by a "random triangle". For instance Kent Morrison finds two other perfectly natural spaces parametrizing the triangles: (1) if we take l to have unit length then we can parametrize all the triangles by pairs of real numbers (thought of as the other two sides of the triangle) (a, b) satisfying $1 - b \leq a \leq b \leq 1$, which describes a triangular region in the ab-plane; (2) the pairs of real numbers (thought of the angles at A and B) (α, β) satisfying $0 \leq \alpha \leq \beta \leq \pi - \alpha - \beta$. As an exercise you might see if you can find the probabilities in these two spaces (answers: $\approx 57.08\%$ and 75% respectively). You can see his full solutions on the Puzzle of the Week webpage.