Puzzle of the Week
Feb 20 - 26, 2009

Fix a line segment \( l \) in the plane. Among all triangles which have \( l \) as longest side one is chosen at random. What is the probability it is obtuse?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week’s email announcement. Anybody is welcome to make a submission.

**Solution:** I found this problem in Lewis Carroll’s “Pillow Problems” (loaned to me by Joe Borzellino) where the statement simply read “what is the probability a random triangle is obtuse”. Unfortunately there are many possible interpretations of a “random triangle” (see the last paragraph) so I included the fixed line segment \( l \) to try and corral people into the following solution:

Since \( l \) is fixed in the plane a triangle having \( l \) as a side is determined by the location of the opposite vertex \( C \). Let the endpoints of \( l \) be denoted \( A \) and \( B \). There is no loss in generality if we take the length of \( l \) to be 2. Then if \( l \) is the longest side of the triangle \( ABC \) it must be that \( C \) lies within the circle of radius 2 centered at \( A \) (so that \(|AC|\) does not exceed 2) and similarly within the circle of radius 2 centered at \( B \). Call this region \( R \) and denote the points of intersection of these two circles by \( P, Q \) (see the figure below). Then \( R \) has area:

\[
\{\text{area of sectors } PAQ\} + \{\text{area of sector } PBQ\} - \{\text{area of triangles } ABP, ABQ\}
\]

Notice that triangle \( ABP \) (as well as triangle \( ABQ \)) is equilateral with side length 2. In particular it follows that angle \( PAQ \) (as well as \( PBQ \)) is \( \frac{2\pi}{3} \) and that each triangle has area (by Heron’s formula, for instance) \( \sqrt{3} \). So the region \( R \) of all possible locations for the vertex \( C \) has area

\[
2 \times \left\{ \frac{\pi}{3} 2^2 \right\} - 2 \times \left\{ \sqrt{3} \right\} = \frac{8\pi}{3} - 2\sqrt{3}
\]
Now given a vertex $C$ within the region $R$ the angles at vertices $A$ and $B$ will be $< \frac{\pi}{2}$. So the triangle will be obtuse if and only if the angle $\theta$ at vertex $C$ is $> \frac{\pi}{2}$. By the Law of Cosines we find

$$\cos \theta = \frac{|AC|^2 + |BC|^2 - |AB|^2}{2|AC||BC|} < 0 \iff |AC|^2 + |BC|^2 < |AB|^2$$

Establish coordinates so that $l$ lies horizontally centered at the origin and let $C = (x, y)$. Then the above inequality becomes:

$$\{(x + 1)^2 + y^2\} + \{(x - 1)^2 + y^2\} < 4$$

which reduces to $x^2 + y^2 < 1$. So the triangle is obtuse iff $C$ lies inside the circle with diameter $l$, which has area $\pi$ (see the figure below). The probability a random triangle is obtuse is therefore the ratio of this area to the area of the region $R$, which gives the formula

$$\frac{3\pi}{8\pi - 6\sqrt{3}} \approx 63.94\%.$$

It should be noted that one can still argue about what is meant by a “random triangle”. For instance Kent Morrison finds two other perfectly natural spaces parametrizing the triangles: (1) if we take $l$ to have unit length then we can parametrize all the triangles by pairs of real numbers (thought of as the other two sides of the triangle) $(a, b)$ satisfying $1 - b \leq a \leq b \leq 1$, which describes a triangular region in the $ab$-plane; (2) the pairs of real numbers (thought of the angles at $A$ and $B$) $(\alpha, \beta)$ satisfying $0 \leq \alpha \leq \beta \leq \pi - \alpha - \beta$. As an exercise you might see if you can find the probabilities in these two spaces (answers: $\approx 57.08\%$ and $75\%$ respectively). You can see his full solutions on the Puzzle of the Week webpage.