Cal Poly Department of Mathematics

Puzzle of the Week

Feb 13 - 19, 2009

Suggested by Kent Morrison:

Fix three points in the plane at $(1,0),(-1,0),(0,h)$ where $h>0$. Find the point(s) (x,y) the sum of whose distances to these three points is minimized.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: The point is given by

$$P=(0,t) \text{ where } t=\min\left\{h,\frac{1}{\sqrt{3}}\right\}.$$

Note that for $h>3^{-\frac{1}{2}}$ this point is independent of h!

Most solutions begin "clearly P lies on the (positive) y-axis...". I can't really argue with that, but I will include a geometric argument to justify this anyway. (Jeff Liese gave a different argument, using Lagrange multipliers, which proves that of all triangles with fixed base and height the isosceles minimizes the perimeter).

Let $F_1=(-1,0)$, $F_2=(1,0)$, $H=(0,h)$ and let P be a point which minimizes the sum of the distances to these points. Consider the (unique) ellipse passing through P and having F_1,F_2 as foci. By its definition every point on this ellipse has the same sum of distances to F_1,F_2. Now if H lies inside this ellipse, then by comparing with the smaller ellipse passing through H we see that replacing P with H gives a smaller sum of distances to F_1,F_2,H, a contradiction. Otherwise we note that the point on the ellipse intersecting the positive y axis is the closest to H and therefore P must be this point.

Now write $P = (0, t)$ with $t \in [0, h]$ and set

$$g(t) = h - t + 2\sqrt{1 + t^2}\,, \quad 0 \le t \le h.$$

We consider the critical points of g:

$$g'(t) = -1 + \frac{2t}{\sqrt{1+t^2}} = 0 \iff t = \pm\frac{1}{\sqrt{3}}$$

So if $h \ge 1/\sqrt{3}$ then we have one critical point on $[0, h]$, namely $t = 1/\sqrt{3}$, and since $g''(1/\sqrt{3}) > 0$ this is the unique minimum.

On the other hand if $0 < h < 1/\sqrt{3}$ then there are no critical points so we must test the end points:

$$g(0) = h + 2, \quad g(h) = 2\sqrt{1 + h^2}.$$

By squaring both sides and doing a little algebra we find $g(h) \le g(0)$ when $h < 1/\sqrt{3}$. Hence our final answer.