Suggested by Kent Morrison:

Fix three points in the plane at $(1, 0), (-1, 0), (0, h)$ where $h > 0$. Find the point(s) $(x, y)$ the sum of whose distances to these three points is minimized.

*Solution:* The point is given by

$$P = (0, t)$$

where $t = \min \left\{ h, \frac{1}{\sqrt{3}} \right\}$. 

Note that for $h > 3^{-\frac{1}{2}}$ this point is independent of $h$!

Most solutions begin “clearly $P$ lies on the (positive) $y$-axis...”. I can’t really argue with that, but I will include a geometric argument to justify this anyway. (Jeff Liese gave a different argument, using Lagrange multipliers, which proves that of all triangles with fixed base and height the isosceles minimizes the perimeter).

Let $F_1 = (-1, 0), F_2 = (1, 0), H = (0, h)$ and let $P$ be a point which minimizes the sum of the distances to these points. Consider the (unique) ellipse passing through $P$ and having $F_1, F_2$ as foci. By its definition every point on this ellipse has the same sum of distances to $F_1, F_2$. Now if $H$ lies inside this ellipse, then by comparing with the smaller ellipse passing through $H$ we see that replacing $P$ with $H$ gives a smaller sum of distances to $F_1, F_2, H$, a contradiction. Otherwise we note that the point on the ellipse intersecting the positive $y$ axis is the closest to $H$ and therefore $P$ must be this point.
Now write $P = (0, t)$ with $t \in [0, h]$ and set

$$g(t) = h - t + 2\sqrt{1 + t^2}, \quad 0 \leq t \leq h.$$ 

We consider the critical points of $g$:

$$g'(t) = -1 + \frac{2t}{\sqrt{1 + t^2}} = 0 \iff t = \pm \frac{1}{\sqrt{3}}$$

So if $h \geq 1/\sqrt{3}$ then we have one critical point on $[0, h]$, namely $t = 1/\sqrt{3}$, and since $g''(1/\sqrt{3}) > 0$ this is the unique minimum.

On the other hand if $0 < h < 1/\sqrt{3}$ then there are no critical points so we must test the end points:

$$g(0) = h + 2, \quad g(h) = 2\sqrt{1 + h^2}.$$ 

By squaring both sides and doing a little algebra we find $g(h) \leq g(0)$ when $h < 1/\sqrt{3}$. Hence our final answer.