Cal Poly Department of Mathematics

Puzzle of the Week

Jan 30 - Feb 5, 2009

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that

$$\lim_{x\to\infty} (f(x) + f'(x)) = 0 \implies \lim_{x\to\infty} f(x) = 0.$$

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

<u>*Solution:*</u> Note that at points of relative extrema the derivative f' vanishes. So the key idea, for me at least, is that if f has relative extrema for all large enough values of x then we can argue that if $f + f'$ is small then f has to be as well. To that end we consider two separate cases: (1) the sign of $f'(x)$ does not change after some point; (2) $f'(x)$ changes sign indefinitely. What follows is a rigorous, though perhaps wordy, proof.

Suppose first that $f'(x)$ does not change sign for all large enough values of x, say $f'(x) \geq 0$ for every $x > N$. Then f is eventually increasing and so either approaches a finite limit or $+\infty$. If this limit is zero we are done. If $f(x)$ approaches a positive limit, or $+\infty$, then for large enough x we can bound $f(x)$ away from 0. Since f' is positive this contradicts $f(x) + f'(x) \to 0$. Hence $f(x) \to -L$ for some $L > 0$ and it must follow that $f'(x) \to L$. Choose $N' \geq N$ large enough that for $x \geq N'$ we have $|f(x) + L| < \frac{L}{2}$ and $|f'(x) - L| < \frac{L}{2}$. Pick any $x > N'$ and apply the Mean Value Theorem to f with x and $x + 2$; we find:

$$\exists \alpha > N' \ : \ 0 \leq f'(\alpha) < \frac{L}{2}$$

contradicting our choice of N'.

In a similar fashion we see $f'(x)$ cannot be ≤ 0 for all large values of x.

Now suppose that $f'(x)$ changes sign indefinitely. Suppose, by contradiction, that $f(x) \not\to 0$. Then there is an $\epsilon > 0$ such that beyond any given $N \in \mathbb{R}$ we can find an x for which $|f(x)| > \epsilon$. Replacing f by $-f$ if necessary we may assume we can always find such x with $f(x) > \epsilon$. Choose N so that $x > N \implies |f(x) + f'(x)| < \epsilon$. Now, if $f(x) > \epsilon$ for *all* large values of x then we get a contradiction by choosing $x > N$ with $f'(x) > 0$, since for this x, $f(x) + f'(x) > \epsilon$. Hence the graph $y = f(x)$ crosses the line $y = \epsilon$ for arbitrarily large values of x. Thus we can find $N < a < b < c$ such that:

$$f(a) = f(c) = \epsilon, \quad \text{and} \quad f(b) > \epsilon.$$

Since f is continuous it attains a maximum on $[a, c]$, say at x_0. Since f is differentiable it must be that $f'(x_0) = 0$. But then

$$f(x_0) + f'(x_0) \geq f(b) + 0 > \epsilon$$

contradicting $x_0 > N$.

Source: G. H. Hardy ,"A Course of Pure Mathematics"