Cal Poly Department of Mathematics

Puzzle of the Week

Jan 16 - 22, 2009

Three circles in the plane are each tangent to a given line (at distinct points) as well as mutually tangent to each other. Find the radius of the smallest circle as a function of the radii of the other two.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

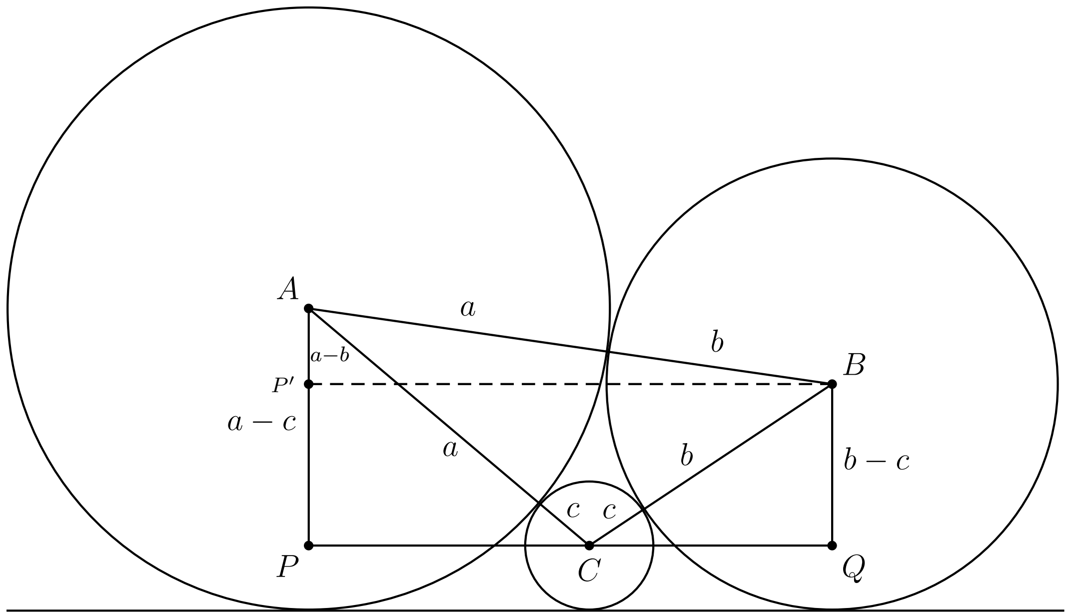
before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: If a, b are the radii of the two larger circles, c the radius of the smallest, then

$$c = \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$

A few people gave solutions to the special case when the two largest circles have the same radius, unfortunately the problem statement did not make this assumption!

One solution places the line horizontally and positions the circles above it. Consider the diagram on the next page:



From the picture we have $|PC|+|CQ| = |P'B|$. By the pythagorean theorem (three times) we find

$$2\sqrt{ac} + 2\sqrt{bc} = 2\sqrt{ab}.$$

Solving this equation for c leads to the solution above.

Jonathan Shapiro points out to me that Descarte's Theorem on four mutually tangent circles applies: If four circles have radii $r_1 \leq r_2 \leq r_3 \leq r_4$ (say) and are mutually tangent then $\left(\sum r_i^{-1}\right)^2 = 2\left(\sum r_i^{-2}\right)$. [There is a sign issue here, depending on how the circles are situated, but I'll ignore this here.] Letting $r_4 \to \infty$ (the largest circle becomes a line) and solving for r_1 recovers the formula above.