Physics 403 HW#9 Solutions:

7.18: a) $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$, $J^\nu = (\rho, \vec{J})$

Let $\nu = 0$ $\quad \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0 = \frac{4\pi}{c}\rho$, $\partial_\mu = (\partial_0, \vec{\nabla}) = \left(\frac{\partial}{c\partial t}, \vec{\nabla}\right)$

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c}\rho$$

$$0 + \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = \frac{4\pi}{c}\rho$$

or $\boxed{(\vec{\nabla}\cdot\vec{E}) = \frac{4\pi}{c}\rho}$

For spatial index $\nu = i = 1,2,3$

$$\partial_\mu F^{\mu i} = \frac{4\pi}{c} J^i$$

$$\partial_0 F^{0i} + \partial_1 F^{1i} + \partial_2 F^{2i} + \partial_3 F^{3i} = \frac{4\pi}{c} J^i$$

x-component, $i=1$, $F^{01} = -E_x$, $F^{11} = 0$, $F^{21} = B_z$, $F^{31} = -B_y$, $J^1 = J_x$

$$\frac{\partial}{c\partial t}(-E_x) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y = \frac{4\pi}{c} J_x$$

$$-\frac{1}{c}\frac{\partial E_x}{\partial t} + (\vec{\nabla}\times\vec{B})_x = \frac{4\pi}{c} J_x$$

(i.e. x-component of $(\vec{\nabla}\times\vec{B}) - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$). A similar result holds for y & z. w/ $i = 2,3$ respectively

b) $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

$$\partial_\nu \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\partial_\nu J^\nu = 0$$

To see that $\partial_\nu\partial_\mu F^{\mu\nu} = 0$, we know $F^{\mu\nu} = -F^{\nu\mu}$

<u>but</u> $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = -\partial_\mu\partial_\nu F^{\nu\mu}$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu} \quad \text{(relabeling } \mu\to\nu,\ \nu\to\mu)$$

$$\Rightarrow 2\partial_\nu\partial_\mu F^{\mu\nu} = 0$$

or $\partial_\nu\partial_\mu F^{\mu\nu} = 0$

And we have the charge continuity equation, $\partial_\nu J^\nu = 0$ as expected!

7.21: $\lambda = i\hbar K a e^{-\frac{i}{\hbar} p\cdot x}$, p is 4-momentum of photon

$\Rightarrow p = \left(\frac{E}{c}, \vec{p}\right)$ w/ $p\cdot p = 0$

or $\frac{E^2}{c^2} - \vec{p}^{\,2} = 0$

$\Rightarrow E = c|\vec{p}|$

a) $\Box^2 \lambda = 0$

$\partial_\mu \partial^\mu \lambda = 0$

w/ $\partial_\mu = \left(\frac{\partial}{c\partial t}, \vec{\nabla}\right)$ & $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$\partial^\mu \lambda = i\hbar a K \partial^\mu e^{-\frac{i}{\hbar} p\cdot x}$, $p\cdot x = \left(\frac{E}{c}, \vec{p}\right)\cdot(ct, \vec{x})$

$= \frac{E}{c} ct - \vec{p}\cdot\vec{x}$

$p\cdot x = (Et - \vec{p}\cdot\vec{x})$

Look at $\partial^0 \lambda$

or $\partial^0 e^{-\frac{i}{\hbar} p\cdot x} = \frac{1}{c}\frac{\partial}{\partial t} e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

$= -\frac{i}{\hbar} e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})} \cdot \frac{E}{c}$

$\Rightarrow \partial^0 \lambda = i\hbar K a \, \frac{-i}{\hbar} \frac{E}{c} e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

$\partial^0 \lambda = \frac{E}{c} K a e^{-\frac{i}{\hbar} p\cdot x}$

Similarly $\partial^i \lambda = i\hbar K a \, \partial^i e^{-\frac{i}{\hbar} p\cdot x} = i\hbar K a\left(-\vec{\nabla} e^{-\frac{i}{\hbar}(Et - \vec{x}\cdot\vec{p})}\right)$

Look @ x-component of $-\vec{\nabla} e^{-\frac{i}{\hbar} x\cdot p}$

$-\frac{\partial}{\partial x} e^{-\frac{i}{\hbar}(Et - xp_x - yp_y - zp_z)} = -\left(\frac{-i}{\hbar}\right) e^{-\frac{i}{\hbar} p\cdot x} \cdot -p_x = \frac{-i}{\hbar} p_x e^{-\frac{i}{\hbar} p\cdot x}$

Similarly for y & z

$\Rightarrow \partial^i \lambda = i\hbar K a \frac{-i}{\hbar} p_i = K a e^{-\frac{i}{\hbar} p\cdot x} p^i$

$\therefore \partial^\mu \lambda = \left(\frac{E}{c}, \vec{p}\right) \frac{\lambda}{i\hbar}$

Similarly $\partial_\mu \lambda = \left(\frac{1}{c}\frac{\partial}{\partial t}, +\vec{\nabla}\right) = \left(\frac{E}{c}, -\vec{p}\right) \frac{\lambda}{i\hbar}$

$\Rightarrow \partial_\mu \partial^\mu \lambda = \underbrace{\left(\frac{E^2}{c^2} - \vec{p}^{\,2}\right)}_{0} \frac{\lambda}{(i\hbar)^2} = 0$ as desired

b) $A^{\mu'} \longrightarrow A^\mu + \partial^\mu \lambda$

Now our Lorentz Condition

$$\partial_\mu A'^\mu = 0 \quad \text{becomes}$$

$$\partial_\mu A^\mu + \partial_\mu \partial^\mu \lambda = 0 = \partial_\mu A^\mu$$

Since $\partial_\mu \partial^\mu \lambda = 0$ as shown in a)

Previously $A^\mu(x) = a e^{-\frac{i}{\hbar} x \cdot p} \epsilon^\mu(p)$

w/ $A^{\mu'}(x) = a e^{-\frac{i}{\hbar} x \cdot p} \epsilon^\mu + \partial^\mu i\hbar K a e^{-\frac{i}{\hbar} p \cdot x}$

as shown in <u>detail</u> in a) $i\hbar \partial^\mu e^{-\frac{i}{\hbar} p \cdot x} = p^\mu e^{-\frac{i}{\hbar} p \cdot x}$

$\Rightarrow A^{\mu'}(x) = a e^{-\frac{i}{\hbar} p \cdot x} + K a e^{-\frac{i}{\hbar} p \cdot x} p^\mu$

or $A'^\mu(x) = a e^{-\frac{i}{\hbar} p \cdot x} (\epsilon^\mu + K p^\mu)$

$\Rightarrow \epsilon^{\mu'} = \epsilon^\mu + K p^\mu$

And as Griffith notes, if we choose

$$K = -\frac{\epsilon^0}{p^0}$$

then $\epsilon^{0'} = 0$ and our Coulomb gauge is satisfied!

7.22: Show $\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \gamma^\mu p_\mu + mc$

with $(u\bar{u})_{ij} \equiv u_i \bar{u}_j = u_i (u^\dagger \gamma_0)_j$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & -u_3^* & -u_4^* \end{pmatrix}$$

$$(u\bar{u})_{ij} = \begin{pmatrix} u_1 u_1^* & u_1 u_2^* & -u_1 u_3^* & -u_1 u_4^* \\ u_2 u_1^* & u_2 u_2^* & -u_2 u_3^* & -u_2 u_4^* \\ u_3 u_1^* & u_3 u_2^* & -u_3 u_3^* & -u_3 u_4^* \\ u_4 u_1^* & u_4 u_2^* & -u_4 u_3^* & -u_4 u_4^* \end{pmatrix}$$

For $u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{c p_z}{E + mc^2} \\ \frac{c(p_x + i p_y)}{E + mc^2} \end{pmatrix}$ w/ $N = \sqrt{|E| + mc^2 / c}$

$$\therefore \quad u^{(1)} \bar{u}^{(1)} = N^2 \begin{pmatrix} 1 & 0 & \frac{-c p_z}{E+mc^2} & -\frac{c(p_x - i p_y)}{E+mc^2} \\ 0 & 0 & 0 & 0 \\ \frac{c p_z}{E+mc^2} & 0 & \frac{-c^2 p_z^2}{(E+mc^2)^2} & \frac{-c^2 p_z (p_x - i p_y)}{(E+mc^2)^2} \\ \frac{c(p_x + i p_y)}{E+mc^2} & 0 & \frac{-c^2 p_z (p_x + i p_y)}{(E+mc^2)^2} & \frac{-c^2 (p_x^2 + p_y^2)}{(E+mc^2)^2} \end{pmatrix}$$

Similarly w/

$$u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

$$\Rightarrow u^{(2)}\bar{u}^{(2)} = N^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{+cp_z}{E+mc^2} \\ 0 & \frac{c(p_x+ip_y)}{E+mc^2} & \frac{-c^2(p_x^2+p_y^2)}{(E+mc^2)^2} & \frac{c^2p_z(p_x-ip_y)}{(E+mc^2)^2} \\ 0 & \frac{cp_z}{E+mc^2} & \frac{c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{-c^2p_z^2}{(E+mc^2)^2} \end{pmatrix}$$

$$\therefore \sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = N^2 \begin{pmatrix} 1 & 0 & \frac{-cp_z}{E+mc^2} & \frac{-c(p_x-ip_y)}{E+mc^2} \\ 0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2} \\ \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2} & \frac{-c^2\bar{p}^2}{(E+mc^2)^2} & 0 \\ \frac{c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2} & 0 & \frac{-c^2\bar{p}^2}{(E+mc^2)^2} \end{pmatrix}$$

and w/ $N^2 = \frac{E+mc^2}{c}$

$$\Rightarrow \sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \begin{pmatrix} \frac{E}{c}+mc & 0 & -p_z & -(p_x-ip_y) \\ 0 & \frac{E}{c}+mc & -(p_x+ip_y) & p_z \\ p_z & p_x-ip_y & \frac{-c\bar{p}^2}{E+mc^2} & 0 \\ p_x+ip_y & p_z & 0 & \frac{-c\bar{p}^2}{E+mc^2} \end{pmatrix}$$

7.22 (cont.): Now working with the other side of the equation $(\gamma^\mu p_\mu + mc) = \gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3 + mc$

$$(\gamma^\mu p_\mu + mc) = \begin{pmatrix} \frac{E}{c} & 0 & 0 & 0 \\ 0 & E/c & 0 & 0 \\ 0 & 0 & -E/c & 0 \\ 0 & 0 & 0 & -\frac{E}{c} \end{pmatrix} - \begin{pmatrix} 0 & \bar{\sigma}\cdot\bar{p} \\ -\bar{\sigma}\cdot\bar{p} & 0 \end{pmatrix} + \begin{pmatrix} mc & 0 \\ 0 & mc \end{pmatrix}$$

$$(\gamma^\mu p_\mu + mc) = \begin{pmatrix} E/c + mc & 0 \\ 0 & -\frac{E}{c} + mc \end{pmatrix} - \begin{pmatrix} 0 & \sigma_x p_x + \sigma_y p_y + \sigma_z p_z \\ -(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z & 0 \end{pmatrix}$$

and $\sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix} 0 & p_x \\ p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ip_y \\ ip_y & 0 \end{pmatrix} + \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$

$$\therefore (\gamma^\mu p_\mu + mc) = \begin{pmatrix} \frac{E}{c} + mc & 0 & -p_z & -(p_x - ip_y) \\ 0 & \frac{E}{c} + mc & -(p_x + ip_y) & p_z \\ p_z & p_x - ip_y & -\left(\frac{E}{c} - mc\right) & 0 \\ p_x + ip_y & -p_z & 0 & -\left(\frac{E}{c} - mc\right) \end{pmatrix}$$

Which is the <u>same</u> as what we have at the bottom of pg. 5 since

$$\bar{p}^2 = \frac{E^2}{c^2} - m^2c^2$$

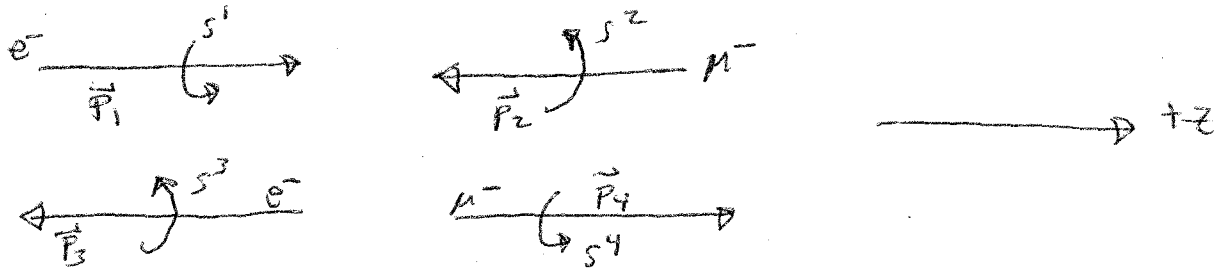
$$\bar{p}^2 = \left(\frac{E}{c} + mc\right)\left(\frac{E}{c} - mc\right)$$

$$\Rightarrow \frac{-c\bar{p}^2}{E + mc^2} = -\left(\frac{E}{c} - mc\right)$$

7.24: For Helicity refresher see fig. 4.9, pg. 124

Helicity +1 $\Rightarrow$ spin and velocity are parallel 

$$\mathcal{M} = \frac{-g_e^2}{(p_1-p_3)^2}\left[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\right]\left[\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)\right]$$



$p_1 = \left(\frac{E_1}{c}, \vec{p}_1\right) \qquad p_2 = \left(\frac{E_2}{c}, \vec{p}_2\right)$ in c.m. $\vec{p}_1 = -\vec{p}_2$

$p_3 = \left(\frac{E_3}{c}, \vec{p}_3\right) \qquad p_4 = \left(\frac{E_4}{c}, \vec{p}_4\right)$ in c.m. $\vec{p}_3 = -\vec{p}_4$

$\Rightarrow |\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4|$ and $\frac{E_1}{c} = \frac{E_3}{c}$ and $\frac{E_2}{c} = \frac{E_4}{c}$ w/ $\frac{E}{c} = \sqrt{\vec{p}^2 + m^2c^2}$

$p_1 - p_3 = \left(\frac{E_1}{c}, \vec{p}_1\right) - \left(\frac{E_1}{c}, -\vec{p}_1\right) = (0, 2\vec{p}_1)$

$\Rightarrow (p_1-p_3)^2 = -4\vec{p}_1$ w/ $\vec{p}_1^{\,2} = \frac{E_1^2}{c^2} - m_e^2c^2$, m_e = mass of e^-

$$u^{(s_1)}(p_1) = u(1) = \begin{pmatrix} \sqrt{(E_1+m_ec^2)/c} \\ 0 \\ \sqrt{(E_1-m_ec^2)/c} \\ 0 \end{pmatrix} \qquad u(4) = \begin{pmatrix} \sqrt{(E_2+m_\mu c^2)/c} \\ 0 \\ \sqrt{(E_2-m_\mu c^2)/c} \\ 0 \end{pmatrix}$$

m_μ = mass of μ^-

(See problem 7.6)

$$u(2) = \begin{pmatrix} 0 \\ \sqrt{(E_2+m_\mu c^2)/c} \\ 0 \\ \sqrt{(E_2-m_\mu c^2)/c} \end{pmatrix} \qquad u(3) = \begin{pmatrix} 0 \\ \sqrt{(E_1+m_e c^2)/c} \\ 0 \\ \sqrt{(E_1-m_e c^2)/c} \end{pmatrix}$$

note the sign in the 4^{th} component of the $u(2)$ & $u(3)$ spinors, since they are moving in the $(-)$ z direction, $-p_z = +p_z$.

Look at $\left[\bar{u}(3)\gamma^{\mu}u(1)\right]\left[\bar{u}(4)\gamma_{\mu}u(2)\right]$

$$=\left[\bar{u}(3)\gamma^{0}u(1)\right]\left[\bar{u}(4)\gamma_{0}u(2)\right]-\left[\bar{u}(3)\gamma^{i}u(1)\right]\left[\bar{u}(4)\gamma_{i}u(2)\right]$$, Sum over i implied

$$= u^{\dagger}(3)u(1)\,u^{\dagger}(4)u(2) - \left[u^{\dagger}(3)\gamma^{0}\gamma^{i}u(1)\right]\left[u^{\dagger}(4)\gamma^{0}\gamma_{i}u(2)\right]$$
(Since $(\gamma^0)^2 = \mathbb{1}$)

$$= 0 - \left[\begin{pmatrix} 0 & \sqrt{(E_1+m_ec^2)/c} & 0 & \sqrt{(E_1-m_ec^2)/c}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\gamma_i\begin{pmatrix}\sqrt{(E_1+m_ec^2)/c}\\0\\\sqrt{(E_1-m_ec^2)/c}\\0\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}\sqrt{(E_2+m_\mu c^2)/c} & 0 & \sqrt{(E_2-m_\mu c^2)/c} & 0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\gamma^i\begin{pmatrix}0\\\sqrt{(E_2+m_\mu c^2)/c}\\0\\\sqrt{(E_2-m_\mu c^2)/c}\end{pmatrix}\right]$$

$$= -\begin{pmatrix}0 & \sqrt{(E_1+m_ec^2)/c} & 0 & -\sqrt{(E_1-m_ec^2)/c}\end{pmatrix}\gamma_i\begin{pmatrix}\sqrt{\frac{E_1+m_ec^2}{c}}\\0\\\sqrt{\frac{E_1-m_ec^2}{c}}\\0\end{pmatrix}\cdot\begin{pmatrix}\sqrt{\frac{E_2+m_\mu c^2}{c}} & 0 & -\sqrt{\frac{E_2-m_\mu c^2}{c}} & 0\end{pmatrix}\cdot\gamma^i\cdot\begin{pmatrix}0\\\sqrt{\frac{E_2+m_\mu c^2}{c}}\\0\\\sqrt{\frac{E_2-m_\mu c^2}{c}}\end{pmatrix}$$

Now

$$\gamma_i = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} \quad \text{or} \quad \gamma_1 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix} \qquad \gamma_3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$$

Look at $i=1$ term (Matrix multiplication by γ_1 has been done)

$$-\left(0 \;\; \sqrt{\frac{E_1+m_ec^2}{c}} \;\; 0 \;\; -\sqrt{\frac{E-m_ec^2}{c}}\right)\begin{pmatrix} 0 \\ \sqrt{\frac{E_1-m_ec^2}{c}} \\ 0 \\ -\sqrt{\frac{E_1+m_ec^2}{c}}\end{pmatrix} \cdot \left(\sqrt{\frac{E_2+m_\mu c^2}{c}} \;\; 0 \;\; -\sqrt{\frac{E_2-m_\mu c^2}{c}} \;\; 0\right)\begin{pmatrix} \sqrt{\frac{E_2-m_\mu c^2}{c}} \\ 0 \\ -\sqrt{\frac{E_2+m_\mu c^2}{c}} \\ 0\end{pmatrix}$$

$$= -2\sqrt{\frac{(E_1+m_ec^2)(E_1-m_ec^2)}{c^2}} \cdot 2\sqrt{\frac{(E_2+m_\mu c^2)(E_2-m_\mu c^2)}{c^2}}$$

$$= -4\sqrt{\frac{E_1^2}{c^2}-m_e^2c^2} \cdot \sqrt{\frac{E_2^2}{c^2}-m_\mu^2c^2}$$

$$= -4\,|\vec{p}_1|\,|\vec{p}_2|$$

$$= -4\,\vec{p}_1^{\,2}$$

Look at $i=2$ term (again matrix multiplication by γ_2 has been done)

$$-\left(0 \;\; \sqrt{\frac{E_1+m_ec^2}{c}} \;\; 0 \;\; -\sqrt{\frac{E_1-m_ec^2}{c}}\right)\begin{pmatrix} 0 \\ i\sqrt{\frac{E_1-m_ec^2}{c}} \\ 0 \\ -i\sqrt{\frac{E_1+m_ec^2}{c}}\end{pmatrix} \cdot \left(\sqrt{\frac{E_2+m_\mu c^2}{c}} \;\; 0 \;\; -\sqrt{\frac{E_2-m_\mu c^2}{c}} \;\; 0\right)\begin{pmatrix} -i\sqrt{\frac{E_2-m_\mu c^2}{c}} \\ 0 \\ i\sqrt{\frac{E_2+m_\mu c^2}{c}} \\ 0\end{pmatrix}$$

$$= -2i\sqrt{\frac{E_1^2-m_e^2c^4}{c^2}} \cdot -2i\sqrt{\frac{E_2^2-m_\mu^2c^4}{c^2}}$$

$$= -4\,\vec{p}_1^{\,2}$$

Look @ $i=3$ (after mult. by γ_3)

$$-\left(0 \;\; \sqrt{\frac{E_1+m_ec^2}{c}} \;\; 0 \;\; -\sqrt{\frac{E_1-m_ec^2}{c}}\right)\begin{pmatrix} \sqrt{\frac{E_1-m_ec^2}{c}} \\ 0 \\ -\sqrt{\frac{E_1+m_ec^2}{c}} \\ 0\end{pmatrix} \cdot \;\ldots$$

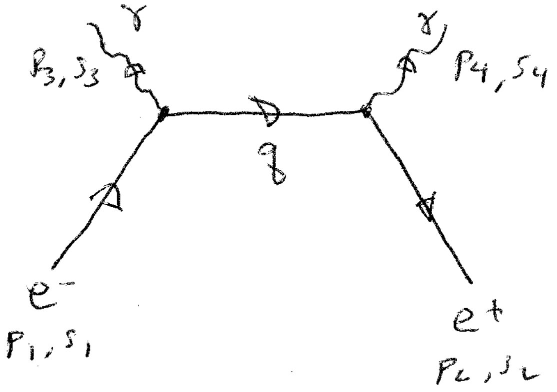
$$= \quad 0 \quad \cdot \quad \ldots \quad = 0\,!$$

$$\mathcal{M} = \frac{-g_e^2}{(p_1 - p_3)^2} \left[\bar{u}(3)\,\gamma^\mu\, u(1)\right]\left[\bar{u}(4)\,\gamma_\mu\, u(2)\right]$$

$$= \frac{-g_e^2}{-4\vec{p}_1^{\,2}} \cdot \left[0 - 4\vec{p}_1^{\,2} - 4\vec{p}_1^{\,2} - 0\right]$$

$$\mathcal{M} = -2g_e^2$$

7.25: $e^+ + e^- \longrightarrow \gamma + \gamma$



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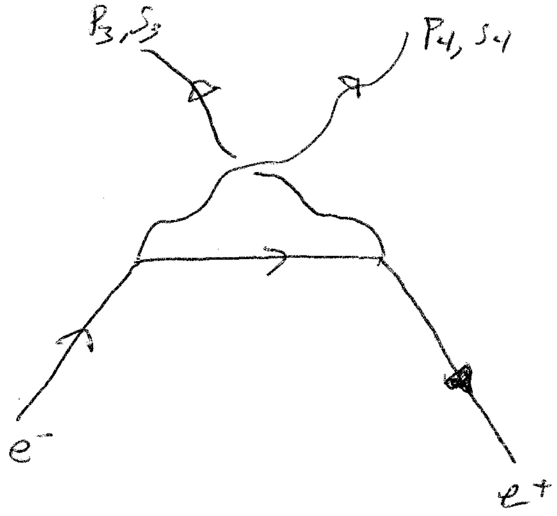

$\mathcal{M}_1$ $\qquad\qquad$ $\mathcal{M}_2$

$$q + p_2 = p_4 \quad \text{or} \quad p_1 = q + p_3$$

$$q = p_4 - p_2 \quad \text{or} \quad q = p_1 - p_3$$

Taking the shortcut w/o $\int d^4q$ and canceling $\delta^4(p_1+p_2-p_3-p_4)$ give

$$-i\mathcal{M}_1 = \bar{v}(p_2)\, i g_e \gamma^\mu \epsilon^*_\mu(4)\, \frac{i(\not{p}_1 - \not{p}_3 + mc)}{(p_1-p_3)^2 - m^2c^2}\, i g_e \gamma^\nu \epsilon^*_\nu(3)\, u(1)$$

or
$$\mathcal{M}_1 = g_e^2 \frac{\bar{v}(2)\, \not{\epsilon}^*(4)(\not{p}_1 - \not{p}_3 + mc)\, \not{\epsilon}^*(3)\, u(1)}{(p_1-p_3)^2 - m^2c^2}$$

For $\mathcal{M}_2$ let $p_3 \longrightarrow p_4$ and $p_4 \longrightarrow p_3$

$$\mathcal{M}_2 = g_e^2 \frac{\bar{v}(2)\, \not{\epsilon}^*(3)(\not{p}_1 - \not{p}_4 + mc)\, \not{\epsilon}^*(4)\, u(1)}{(p_1-p_4)^2 - m^2c^2}$$

and $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$

<u>note:</u> Amplitudes <u>add</u> since our identical particles are bosons and we don't have to worry about the exclusion principle

7.26: Let's do one of the mixed cases

$$\sum_{\text{spins}} \left[\bar{u}(a)\Gamma_1 v(b)\right]\left[\bar{u}(a)\Gamma_2 v(b)\right]^*$$

Look at $\left[\bar{u}(a)\Gamma_2 v(b)\right]^* = \left[u^\dagger(a)\gamma_0\Gamma_2 v(b)\right]^*$

$$= v^\dagger(b)\Gamma_2^\dagger\gamma_0^\dagger u(a)$$

$$= v^\dagger(b)\gamma_0\gamma_0\Gamma_2^\dagger\gamma_0 u(a), \quad \gamma_0^2 = 1 \text{ \& } \gamma_0^\dagger = \gamma_0$$

$\therefore \left[\bar{u}(a)\Gamma_2 v(b)\right]^* = \bar{v}(b)\bar{\Gamma}_2 u(a)$, w/ $\bar{\Gamma}_2 \equiv \gamma_0\Gamma_2^\dagger\gamma_0$

$$\sum_{\text{spins}} \bar{u}(a)\Gamma_1 v(b)\bar{v}(b)\bar{\Gamma}_2 u(a)$$

$= \sum_{\text{spins}} \bar{u}(a)\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2 u(a)$, using $\sum_{\text{spin}} v\bar{v} = \not{p} - mc$

Let $Q_{ij} = \Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2$

Then we have

$$\sum_{\text{spins}} \bar{u}_i(a) Q_{ij} u_j(a)$$

$$= \sum_{\text{spins}} u_j(a)\bar{u}_i(a) Q_{ij}$$

$= (\not{p}_a + mc)_{ji} Q_{ij}$, using $\sum_{\text{spin}} u\bar{u} = \not{p} + mc$

$$= \mathrm{Tr}\left[(\not{p}_a + mc) Q\right]$$

$$= \mathrm{Tr}\left[(\not{p}_a + mc)\Gamma_1(\not{p}_b - mc)\bar{\Gamma}_2\right]$$

7.34: Electron-Electron Scattering

$$\mathcal{M} = \overbrace{\frac{-g_e^2}{(p_1-p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]}^{\mathcal{M}_1}$$

$$\underbrace{+\frac{g_e^2}{(p_1-p_4)^2}\left[\bar{u}(4)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma_\mu u(2)\right]}_{\mathcal{M}_2}$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\text{spins}} \underbrace{|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_1\mathcal{M}_2^*}_{(\mathcal{M}_1+\mathcal{M}_2)(\mathcal{M}_1^*+\mathcal{M}_2^*)}$$

Look at $\langle \mathcal{M}_1\mathcal{M}_2^*\rangle = \frac{1}{4}\sum_{\text{spins}} \mathcal{M}_1\mathcal{M}_2^*$

$$= \frac{-g_e^4}{(p_1-p_3)^2(p_1-p_4)^2}\cdot\frac{1}{4}\sum_{\text{spins}}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(4)\gamma^\nu u(1)\right]^*\left[u(3)\gamma_\nu u(2)\right]^*$$

Focus in on $\sum_{\text{spins}}$

$$\sum_{\text{spins}}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(1)\gamma^\nu u(4)\right]\left[\bar{u}(2)\gamma_\nu u(3)\right]$$

$$\sum_{\text{spins}}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(1)\gamma^\nu u(4)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(2)\gamma_\nu u(3)\right]$$

using $\sum_{\text{spins}} u(1)\bar{u}(1) = \not{p}_1 + mc$, $\sum_{\text{spins}} u\bar{u}(4) = \not{p}_4 + mc$, $\sum_{\text{spins}} u\bar{u}(2) = \not{p}_2 + mc$

Gives

$$\sum_{\text{spins}} \bar{u}(3)\underbrace{\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_4+mc)\gamma_\mu(\not{p}_2+mc)\gamma_\nu}_{Q_{ij}}u(3)$$

$$\sum_{spin} \bar{u}_i(3)\, Q_{ij}\, u_j(3) = \sum_{spin} u_j(3)\,\bar{u}_i(3)\, Q_{ij}$$

$$= (\not{p}_3 + mc)_{ji}\, Q_{ij} = \mathrm{Tr}\left[(\not{p}_3 + mc)Q\right]$$

$$= \mathrm{Tr}\left[(\not{p}_3 + mc)\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_4 + mc)\gamma_\mu(\not{p}_2 + mc)\gamma_\nu\right]$$

In the high energy limit $E \gg mc^2$, $m \approx 0$, giving

$$\mathrm{Tr}\left[\not{p}_3 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu\right]$$

As Griffith notes, in this massless limit, $p_1 + p_2 = p_3 + p_4$, gives

$$(p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$\cancel{p_1^2}^{\,0} + 2p_1\cdot p_2 + \cancel{p_2^2}^{\,0} = \cancel{p_3\cdot p_3}^{\,0} + 2p_3\cdot p_4 + \cancel{p_4\cdot p_4}^{\,0}$$

$$\text{or} \quad p_1\cdot p_2 = p_3\cdot p_4$$

Similarly for $(p_1 - p_3)^2 = (p_4 - p_2)^2$ and $(p_1 - p_4)^2 = (p_3 - p_2)^2$

$$\Rightarrow \quad p_1\cdot p_3 = p_2\cdot p_4 \qquad\qquad \Rightarrow \quad p_1\cdot p_4 = p_2\cdot p_3$$

Using $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$ (pg. 239)

$$\mathrm{Tr}\left[\not{p}_3 \gamma^\mu p_{1\sigma}\gamma^\sigma \gamma^\nu p_{4\rho}\gamma^\rho \gamma_\mu \not{p}_2 \gamma_\nu\right] = -2\,\mathrm{Tr}\left[\not{p}_3\, p_{4\rho}\gamma^\rho\gamma^\nu p_{1\sigma}\gamma^\sigma \not{p}_2 \gamma_\nu\right]$$

$$= -2\,\mathrm{Tr}\left[\not{p}_3\not{p}_4\gamma^\nu\not{p}_1\not{p}_2\gamma_\nu\right]$$

Using $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}$

$$= -2\,\mathrm{Tr}\left[\not{p}_3\not{p}_4\, 4p_1\cdot p_2\right]$$

$$= -8\,\mathrm{Tr}\left[\not{p}_3\not{p}_4\right] p_1\cdot p_2$$

Using $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$$= -32\,(p_3\cdot p_4)(p_1\cdot p_2)$$

$$\therefore \langle \mathcal{M}_1 \mathcal{M}_2^* \rangle = \frac{-g_e^4}{(p_1 - p_3)^2(p_1 - p_4)^2}\cdot\frac{1}{4}\left[-32(p_3\cdot p_4)(p_1\cdot p_2)\right]$$

or $$\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle = \frac{8 g_e^4 (p_1 \cdot p_2)^2}{(p_1 - p_3)^2 (p_1 - p_4)^2}$$

Since $\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle$ is real, $\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle^* = \langle \mathcal{M}_1^* \mathcal{M}_2 \rangle = \langle \mathcal{M}_1 \mathcal{M}_2^* \rangle$

$$\Rightarrow \langle \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1^* \mathcal{M}_2 \rangle = \frac{16 g_e^4 (p_1 \cdot p_2)^2}{(p_1 - p_3)^2 (p_1 - p_4)^2}$$

Note that $\langle |\mathcal{M}_1|^2 \rangle$ is the <u>same</u> as 7.126 w/ $m = M = 0$

$$\Rightarrow \langle |\mathcal{M}_1|^2 \rangle = \frac{8 g_e^4}{(p_1 - p_3)^4} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right]$$

$$\langle |\mathcal{M}_1|^2 \rangle = \frac{8 g_e^4}{(p_1 - p_3)^4} \left[(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2 \right]$$

For $\langle |\mathcal{M}_2|^2 \rangle$ Let $p_3 \to p_4$ and $p_4 \to p_3$

$$\Rightarrow \langle |\mathcal{M}_2|^2 \rangle = \frac{8 g_e^4}{(p_1 - p_4)^4} \left[(p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2 \right]$$

Recall from Ch 3 (HW #3.23), the Mandelstam Variables defined by $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$

(note: I have droped the $1/c^2$ factor in the definition here)

In the massless limit

$$s = 2 p_1 \cdot p_2, \quad t = -2 p_1 \cdot p_3 \quad \text{and} \quad u = -2 p_1 \cdot p_4$$

or $$\frac{s^2}{4} = (p_1 \cdot p_2)^2, \quad \frac{t^2}{4} = (p_1 \cdot p_3)^2 \quad \text{and} \quad \frac{u^2}{4} = (p_1 \cdot p_4)^2$$

$$\therefore \langle \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1^* \mathcal{M}_2 \rangle = \frac{16 g_e^4 \left(\frac{s^2}{4}\right)}{tu} = \frac{4 g_e^4 s^2}{tu}$$

$$\langle |\mathcal{M}_1|^2 \rangle = \frac{8 g_e^4}{t^2}\left(\frac{s^2}{4} + \frac{u^2}{4}\right) = 2 g_e^4 \frac{(s^2+u^2)}{t^2}$$

and $$\langle |\mathcal{M}_2|^2 \rangle = \frac{8 g_e^4}{u^2}\left(\frac{s^2}{4} + \frac{t^2}{4}\right) = 2 g_e^4 \frac{(s^2+t^2)}{u^2}$$

$$\therefore \langle |\mathcal{M}|^2 \rangle = \langle |\mathcal{M}_1|^2 \rangle + \langle |\mathcal{M}_2|^2 \rangle + \langle \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1^* \mathcal{M}_2 \rangle$$

$$\langle |\mathcal{M}|^2 \rangle = 2 g_e^4 \left[\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu} \right]$$

Getting a common denominator

$$\langle |\mathcal{M}|^2 \rangle = 2 g_e^4 \left[\frac{u^2(s^2+u^2) + t^2(s^2+t^2) + 2s^2 tu}{t^2 u^2} \right]$$

$$\langle |\mathcal{M}|^2 \rangle = 2 g_e^4 \left[\frac{u^4 + t^4 + s^2(t+u)^2}{t^2 u^2} \right]$$

Also recall from HW # 3.23 for $A + A \to A + A$ scattering

$s = 4\bar{p}^2$, $t = -2\bar{p}^2(1-\cos\theta)$ and $u = -2\bar{p}^2(1+\cos\theta)$
(for $m=0$)

$$\Rightarrow (t+u)^2 = (-4\bar{p}^2)^2 = (4\bar{p})^2 = s^2$$

and

$$\langle |\mathcal{M}|^2 \rangle = 2 g_e^4 \left[\frac{u^4 + t^4 + s^4}{t^2 u^2} \right]$$

Which is Griffiths result if you convert back to p_1, p_3 and p_4 !!

7.36: $$\left.\frac{d\sigma}{d\Omega}\right)_{c.m} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(E_1+E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{4E^2}$$, since $E_1 = E_2 = E$ and $|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|$ for elastic scattering of identical particles

Using the result for $\langle|\mathcal{M}|^2\rangle$ from 7.34 for unpolarized electrons.

$$\langle|\mathcal{M}|^2\rangle = 2g_e^4\left[\frac{u^4+t^4+s^4}{t^2u^2}\right]$$

w/ $t = -2\vec{p}^2(1-\cos\theta)$, $u = -2\vec{p}^2(1+\cos\theta)$ and $s = 4\vec{p}^2$

$$\Rightarrow tu = 4\vec{p}^4(1-\cos\theta)(1+\cos\theta) = 4\vec{p}^4(1-\cos^2\theta) = 4\vec{p}^4\sin^2\theta$$

$$\Rightarrow (tu)^2 = t^2u^2 = 16\vec{p}^8\sin^4\theta$$

$$u^4 = \left[-2\vec{p}^2(1+\cos\theta)\right]^4 = 16\vec{p}^8(1+\cos\theta)^4$$

Using the binomial theorem

$$(1+\cos\theta)^4 = \binom{4}{4}1 + \binom{4}{3}\cos\theta + \binom{4}{2}\cos^2\theta + \binom{4}{1}\cos^3\theta + \binom{4}{0}\cos^4\theta$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

$$(1+\cos\theta)^4 = 1 + 4\cos\theta + 6\cos^2\theta + 4\cos^3\theta + \cos^4\theta$$

Similarly for t^4

$$t^4 = \left[-2\vec{p}^2(1-\cos\theta)\right]^4 = 16\vec{p}^8(1-\cos\theta)^4$$ w/ $\cos\theta \to -\cos\theta$

$$\Rightarrow t^4 + u^4 = 16\vec{p}^8\left[2 + 12\cos^2\theta + 2\cos^4\theta\right]$$

$$t^4 + u^4 = 32\bar{p}^8 \left[1 + 6(1-\sin^2\theta) + (1-\sin^2\theta)^2\right]$$

$$= 32\bar{p}^8 \left[1 + 6 - 6\sin^2\theta + 1 - 2\sin^2\theta + \sin^4\theta\right]$$

$$t^4 + u^4 = 32\bar{p}^8 \left[8 - 8\sin^2\theta + \sin^4\theta\right]$$

now $s^4 = (4\bar{p}^2)^4 = 8 \cdot 32\bar{p}^8$

$$\Rightarrow t^4 + u^4 + s^4 = 32\bar{p}^8 \left[16 - 8\sin^2\theta + \sin^4\theta\right]$$

$$t^4 + u^4 + s^4 = 32\bar{p}^8 \left[\sin^4\theta - 8\sin^2\theta + 16\right]$$

$$\Rightarrow \langle |M|^2 \rangle = 2g_e^4 \left[\frac{u^4 + t^4 + s^4}{t^2 u^2}\right]$$

$$= 2g_e^4 \frac{32\bar{p}^8 \left[\sin^4\theta - 8\sin^2\theta + 16\right]}{16\bar{p}^8 \sin^4\theta}$$

$$= 4g_e^4 \left(1 - \frac{8}{\sin^2\theta} + \frac{16}{\sin^4\theta}\right)$$

$$\langle |M|^2 \rangle = 4g_e^4 \left(1 - \frac{4}{\sin^2\theta}\right)^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{g_e^4}{E^2} \left(1 - \frac{4}{\sin^2\theta}\right)^2$$

And I am off by a factor of 2!!

Let me know if you find my mistake!