Solutions to Math 141 Sample Final Exam Problems

The final exam will cover Chapters 2, 3, 4, and 5, except sections 2.4, 4.6, and 4.8. This collection of sample questions is longer than the final, which will have approximately 10 questions.

1. Evaluate the derivatives of the following functions:

   (a) $f(x) = \sin\sqrt{x^2+x+1}\cdot\cos\sqrt[3]{x}$
   $f'(x) = \cos\sqrt{x^2+x+1}\cdot\frac{1}{2}(x^2+x+1)^{-1/2}\cdot(2x+1)\cdot\cos\sqrt[3]{x} + \sin\sqrt{x^2+x+1}\cdot(-\sin\sqrt[3]{x})\cdot\frac{1}{3}x^{-2/3}$

   (b) $f(x) = \dfrac{x^2+1}{\cos^3(3x+1)}$
   $f'(x) = \dfrac{2x\cos^3(3x+1) - (x^2+1)\cdot 3\cos^2(3x+1)\cdot(-\sin(3x+1))\cdot 3}{\cos^6(3x+1)}$

   (c) $f(x) = x^3\tan(3x^2) + \sec(1/x)$
   $f'(x) = 3x^2\tan(3x^2) + x^3\sec^2(3x^2)\cdot(6x) + \sec(1/x)\tan(1/x)\cdot(-1/x^2)$

   (d) $f(x) = \displaystyle\int_{-3x}^{\sin x}\frac{1}{1+t^2}\,dt$
   $f'(x) = \dfrac{1}{1+\sin^2 x}\cos x - \dfrac{1}{1+(-3x)^2}(-3)$

2. Find the equation of the tangent line to the curve $y = f(x)$ at the point $(1, 0)$ if $xy^2 + 5y = x^3 - \cos(xy)$.

   Tangent line: $y - y_1 = m(x - x_1)$, where $x_1 = 1$, $y_1 = 0$, and $m = y'$ at $(1, 0)$.
   Implicit differentiation gives $y^2 + x\cdot 2yy' + 5y' = 3x^2 + \sin(xy)\cdot(y + xy')$.
   Rearranging, $y'(2xy + 5 - x\sin xy) = 3x^2 + y\sin xy - y^2$.
   Isolating y', and substituting $x = 1$, $y = 0$ gives $y' = 3/5$. The tangent line is therefore $y = (3/5)(x-1)$.

3. Determine the following limits:

   (a) $\displaystyle\lim_{x\to-2}\frac{x^3+8}{x^2+x-2} = \lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-1)} = \lim_{x\to-2}\frac{(x^2-2x+4)}{(x-1)} = \frac{12}{-3} = -4$

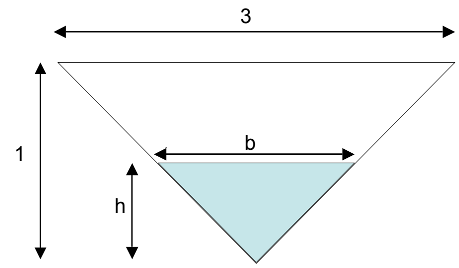
   (b) $\displaystyle\lim_{x\to0}\frac{\sin^2(8x)}{\sin(8x^2)} = \lim_{x\to0}(8x)^2\frac{\sin^2(8x)}{(8x)^2}\cdot\frac{1}{8x^2}\cdot\frac{8x^2}{\sin(8x^2)} = \lim_{x\to0}\frac{(8x)^2}{8x^2}\cdot\left(\frac{\sin(8x)}{8x}\right)^2\frac{8x^2}{\sin(8x^2)} = 8\cdot1^2\cdot1 = 8$

   (c) $\displaystyle\lim_{x\to\infty}\sqrt{x^2+100x}-\sqrt{x^2+50x} = \lim_{x\to\infty}\left(\sqrt{x^2+100x}-\sqrt{x^2+50x}\right)\cdot\frac{\sqrt{x^2+100x}+\sqrt{x^2+50x}}{\sqrt{x^2+100x}+\sqrt{x^2+50x}}$
   $\displaystyle = \lim_{x\to\infty}\frac{(x^2+100x)-(x^2+50x)}{\sqrt{x^2+100x}+\sqrt{x^2+50x}} = \lim_{x\to\infty}\frac{50x}{\sqrt{x^2+100x}+\sqrt{x^2+50x}} = \lim_{x\to\infty}\frac{50}{\sqrt{1+100/x}+\sqrt{1+50/x}}$
   $\displaystyle = \frac{50}{\sqrt{1}+\sqrt{1}} = 25$

   (d) $\displaystyle\lim_{x\to\infty}\frac{3+\sin(7x)}{x} = 0$. Since $-1 \le \sin(7x) \le 1$, we have $\dfrac{2}{x} \le \dfrac{3+\sin(7x)}{x} \le \dfrac{4}{x}$. But $\displaystyle\lim_{x\to\infty}\frac{2}{x} = \lim_{x\to\infty}\frac{4}{x} = 0$, so by the Squeeze Theorem, our limit must also be 0.

4. A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across at the top and have a height of 1 foot. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water rising when the water is 6 inches deep?

Let V be the volume of water in the trough, and h the height or depth of the water. Then we're given $\frac{dV}{dt} = 12$, and asked for $\frac{dh}{dt}$ when $h = 0.5$. Now, $V = 10 \cdot (\text{cross-sectional area}) = 10 \cdot (1/2)bh$, where $b =$ width of surface of water. By similar triangles, $3/1 = b/h$, so $b = 3h$. Therefore $V = 15h^2$. Taking d/dt of V equation gives $dV/dt = 30h \cdot dh/dt$. With $dV/dt = 12$ and $h = 0.5$, this gives $dh/dt = 4/5$ ft/s.



5. Find the area of the largest rectangle with sides parallel to the coordinate axes which can be inscribed in the ellipse $x^2/4 + y^2/9 = 1$.

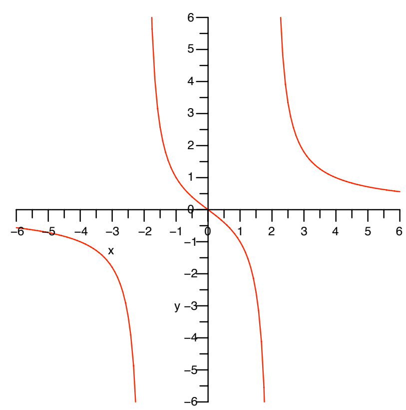
Let (x, y) be the coordinates of the upper right corner of the rectangle. Then the rectangle has width $2x$ and height $2y$, so $A = 4xy$. But (x, y) lies on the ellipse, so satisfies $x^2/4 + y^2/9 = 1$, therefore $y = \sqrt{9 - 9x^2/4}$. Thus, $A = 4x\sqrt{9 - 9x^2/4}$, and

$A' = 4\sqrt{9 - 9x^2/4} + 4x \cdot (1/2)(9 - 9x^2/4)^{-1/2} \cdot (-9x/2) = \dfrac{4(9 - 9x^2/4) - 9x^2}{\sqrt{9 - 9x^2/4}} = \dfrac{18(2 - x^2)}{\sqrt{9 - 9x^2/4}}$

So $A' = 0$ at $x = \sqrt{2}$. It's a local max by the geometry of the problem and since A' is positive for $x < \sqrt{2}$ and negative for $x > \sqrt{2}$. Then $y = \sqrt{9 - 9x^2/4} = 3/\sqrt{2}$, and $A = 4xy = 4\sqrt{2} \cdot 3/\sqrt{2} = 12$.
The area of the largest rectangle is 12.

6. Consider the function $f(x) = \dfrac{3x}{x^2 - 4}$. Then $f'(x) = \dfrac{-3(x^2 + 4)}{(x^2 - 4)^2}$ and $f''(x) = \dfrac{6x(x^2 + 12)}{(x^2 - 4)^3}$. Find all vertical and horizontal asymptotes. Find all local maxima and minima. Describe the intervals where the function is concave up and down. Describe the intervals where the function is increasing and decreasing. Sketch the graph.

Vertical asymptotes: $x = -2$ and $x = 2$ because $\lim_{x \to -2^-} f(x) = -\infty$, $\lim_{x \to -2^+} f(x) = \infty$, $\lim_{x \to 2^-} f(x) = -\infty$, and $\lim_{x \to 2^+} f(x) = \infty$.
Horizontal asymptote: $y = 0$ because $\lim_{x \to -\infty} f(x) = \dfrac{3}{x - 4/x} = 0$, Similarly, $\lim_{x \to \infty} f(x) = 0$.
Local maxima and minima: none! Can only have these at critical points, and there are no critical points because $f'(x) \neq 0$ and $f'(x)$ does not exist only at those points where $f(x)$ itself does not exist.
Concavity: by inspecting the sign of $f''(x)$, $f(x)$ is concave down on $(-\infty, -2) \cup (0, 2)$ and concave up on $(-2, 0) \cup (2, \infty)$. f is decreasing for all x in its domain.



7. State the definition of the derivative. State the definition of continuity.

The derivative $f'(x)$ of $f(x)$ is defined to be $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.

$f(x)$ is defined to be continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

8. Use the definition of the derivative and the identity $\sin(A+B) = \sin A\cos B + \cos A \sin B$ to find the derivative of $f(x) = \sin 5x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(5(x+h)) - \sin 5x}{h} = \lim_{h\to 0}\frac{\sin(5x+5h)-\sin 5x}{h}$$
$$= \lim_{h\to 0}\frac{\sin 5x\cos 5h + \cos 5x \sin 5h - \sin 5x}{h} = \lim_{h\to 0}\left(\sin 5x \frac{\cos 5h - 1}{h} + \cos 5x \frac{\sin 5h}{h}\right)$$
$$= 5\sin 5x \lim_{h\to 0}\left(\frac{\cos 5h - 1}{5h}\right) + 5\cos 5x \lim_{h\to 0}\left(\frac{\sin 5h}{5h}\right) = 5\sin 5x\cdot 0 + 5\cos 5x \cdot 1 = 5\cos 5x$$

9. Show that the equation $\dfrac{x^2(x-1)}{4+5x^2} = 1$ has at least one real solution.

Let $f(x) = \dfrac{x^2(x-1)}{4+5x^2}$. $f(0) = 0 < 1$ and $f(10) = 900/504 > 1$. Since $f(x)$ is continuous, the Intermediate Value Theorem guarantees $f(x) = 1$ for some x, $0 < x < 10$. At that x, the given equation has a solution.

10. Let $f(x) = x^4 + Ax^2 + x$, where A is a constant. Are there values of A for which f has inflection points at both $x = 0$ and $x = 1$?

$f'(x) = 4x^3 + 2Ax + 1$ and $f''(x) = 12x^2 + 2A$. Inflection points are where $f''(x)$ changes sign, which can only happen where $f''(x)$ doesn't exist or where $f''(x) = 0$. But $f''(x)$ exists for all x, so the only candidates for inflection points are where $f''(x) = 0$. If $f''(1) = 0$, then $A = -6$, and $f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$, so $x = 1$ is indeed an inflection point. But then $x = 0$ is not an inflection point. There are no values of A for which f has inflection points at both $x = 0$ and $x = 1$.

In fact, there are no values of A for which f has an inflection point at $x = 0$ at all! If $f''(0) = 0$, then $A = 0$ and $f''(x) = 12x^2$. But then $f''(x)$ doesn't change sign at $x = 0$, so $x = 0$ is not an inflection point.

11. Perform the required integrations:

(a) $\displaystyle\int 6\sec^2 5x\,dx = \frac{6}{5}\tan 5x + C$

(b) $\displaystyle\int \frac{\cos x}{\sqrt{1+2\sin x}}\,dx = \sqrt{1+2\sin x} + C$

(c) $\displaystyle\int \frac{\cos\sqrt{x}}{5\sqrt{x}}\,dx = \frac{2}{5}\sin\sqrt{x} + C$

(d) $\displaystyle\int_0^1 x^5\sqrt[7]{x^3+1}\,dx$

Let $u = x^3+1$. Then $du = 3x^2\,dx$, so $\frac{1}{3}du = x^2\,dx$. Note that $x^3 = u-1$. Then $x^5\,dx = x^3x^2\,dx = (u-1)\frac{1}{3}\,du$. So the integrand transforms to $\sqrt[7]{u}(u-1)\frac{1}{3}\,du$. Also need to change the limits: when $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. The integral becomes

$$\int_1^2 u^{1/7}(u-1)\frac{1}{3}\,du = \frac{1}{3}\int_1^2 u^{8/7} - u^{1/7}\,du = \frac{1}{3}\left(\frac{7}{15}u^{15/7} - \frac{7}{8}u^{8/7}\right)\Big|_1^2 = \frac{1}{3}\left(\frac{7}{15}2^{15/7} - \frac{7}{8}2^{8/7} - \frac{7}{15} + \frac{7}{8}\right)$$

12. Consider the area under the curve $y = x + x^3$ between $x = 0$ and $x = 2$.

(a) Estimate the area using 4 subintervals and a right endpoint approximation.

Area $\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$, where $\Delta x = (b-a)/n = 2/4 = 1/2$, and $x_i = a + i\Delta x$, that is, $x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}$, and $x_4 = 2$. Thus, Area $\approx \frac{1}{2}\left(\frac{1}{2} + \left(\frac{1}{2}\right)^3 + 1 + 1^3 + \frac{3}{2} + \left(\frac{3}{2}\right)^3 + 2 + 2^3\right)$

(b) Find the exact area by setting up a Riemann sum and taking an appropriate limit. You may need the formulas $\displaystyle\sum_{i=1}^n i = \frac{1}{2}n(n+1), \sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{i=1}^n i^3 = \frac{1}{4}n^2(n+1)^2.$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^n\left(\frac{2i}{n} + \left(\frac{2i}{n}\right)^3\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^n \frac{4i}{n^2} + \frac{16i^3}{n^4}$$
$$= \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^n i + \frac{16}{n^4}\sum_{i=1}^n i^3 = \lim_{n\to\infty}\frac{4}{n^2}\frac{n(n+1)}{2} + \frac{16}{n^4}\frac{n^2(n+1)^2}{4} = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right) + 4\left(1+\frac{1}{n}\right)^2 = 6$$

13. "*If $f(x)$ is continuous at a point, then it is differentiable there.*"
"*If $f(x)$ is differentiable at a point, then it is continuous there.*"
One of these statements is true, and one is false.

(a) Which one is false?
The first one. A continuous function is not necessarily differentiable.

(b) Give an example of a function that demonstrates the statement is not true.
$f(x) = |x|$ is continuous but not differentiable at $x = 0$.

14. Derive the iteration formula for Newton's method by the following two steps:

(Step 1) Write the equation for the tangent line to $f(x)$ at $x = x_n$ in terms of x_n, $f(x_n)$, and $f'(x_n)$.
$y - f(x_n) = f'(x_n)(x - x_n)$

(Step 2) Find the value of x where that tangent line intersects the x-axis. You may assume $f'(x_n) \neq 0$.
$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n) \Rightarrow -\dfrac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n \Rightarrow x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

15. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with radius 50 m.
$V = \dfrac{1}{2}\dfrac{4}{3}\pi r^3$, so $dV = 2\pi r^2\, dr = 2\pi(50)^2 \cdot 5 \cdot 10^{-4} = \frac{5}{2}\pi$ m^3.

16. Let

$$f(x) = \begin{cases} 2x - x^2 & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim\limits_{x\to 0^+} f(x)$	(ii) $\lim\limits_{x\to 0^-} f(x)$	(iii) $\lim\limits_{x\to 0} f(x)$	(iv) $\lim\limits_{x\to 3^-} f(x)$	(v) $\lim\limits_{x\to 3^+} f(x)$	(vi) $\lim\limits_{x\to 3} f(x)$
$= 3$	$= 0$	doesn't exist	$= 0$	$= 0$	$= 0$

(b) Where is f discontinuous?
At $x = 0$ (limit doesn't exist) and at $x = 3$ (limit exists, but function not defined).

17. The velocity of a particle as a function of time is given by $v(t) = 3t^2 + 5t + 7 + 9\sin t$.

(a) Find the acceleration $a(t)$.
$a(t) = 6t + 5 + 9\cos t$

(b) Find the position $s(t)$, given $s(0) = 0$.
$s(t) = t^3 + \frac{5}{2}t^2 + 7t - 9\cos t + C$. $s(0) = 0 \Rightarrow C = 9$, so $s(t) = t^3 + \frac{5}{2}t^2 + 7t - 9\cos t + 9$.