MULTI-DIMENSIONAL SYSTEM: Ship Stability

In this computer simulation we will explore a nonlinear multi-dimensional system. As before these systems are governed by equations of the form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots.., x_n) \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots.., x_n) \\
&. \\
&. \\
\dot{x}_n &= f_n(x_1, x_2, \ldots.., x_n)
\end{aligned}$$

where n is the dimension of the system.

A problem in ship stability.

In this lab we'll be using a simplified model of a ship in rolling seas in order to study the ship’s stability and find out how likely it is that the ship may capsize. We'll use the software *Differential Systems* to sketch the phase plane of the motion and continue our use of `Matlab`. This system is an example of a forced nonlinear oscillator.

This lab is designed to bring home the points that, firstly, although a ship in the sea is a very complicated system, people model it reasonably effectively using a simple (often empirical) model. Secondly, even though the model is simple, the behavior is not. In fact, due to the nonlinearities the ship may capsize even though the sea conditions are far from where one would expect capsize to occur.

Introduction.

In this study we'll consider the simple case of a ship broadside to the waves so that the wave action causes the ship to roll from side to side. Figure 1 shows a ship sitting in still water. There are two forces acting on the ship which try to cause it to "sit up straight". These are the weight and the buoyancy which act together to reduce the angle x. If the ship is displaced to one side it tries to restore itself and oscillates until the damping of the water brings it to rest.

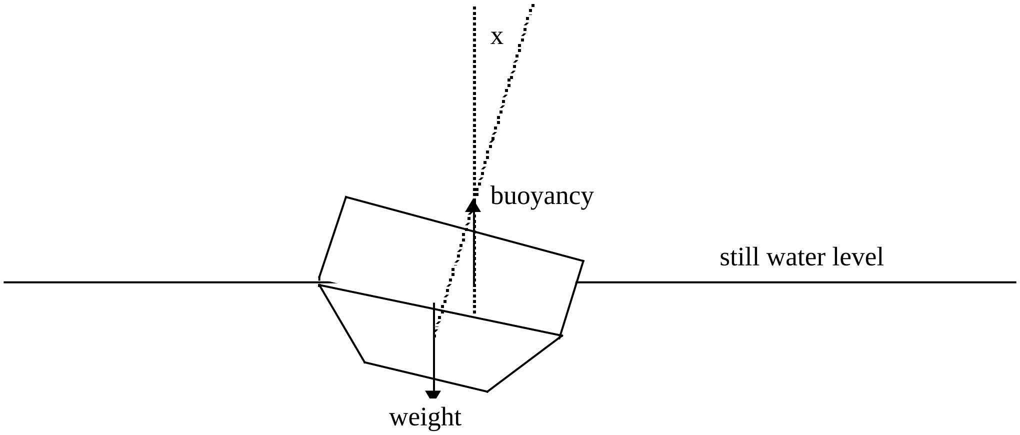

Figure 1: Schematic of ship in still water showing the action of bouyancy and weight to right the ship.

While experiments have deduced the equations of motion for some hull types (you can see these in the references given below) a more general approach is to consider the simplest possible equation that can be investigated interactively. We choose a model which represents a ship that is biased to capsize (this could be due to wind or ice loading or movement of the cargo to one side).

The simplest equation we can have in this case is

$$\ddot{x} + \dot{x}\beta + x - x^2 = F\sin(\omega t) \qquad (1)$$

where x is the angle of inclination of the ship, β is a damping constant, F is the amplitude of the forcing (the size of the waves) and ω is the frequency with which the waves arrive.

<u>Getting a feel for the model (Ignoring the forcing)</u>

In order to see what this equation means we'll first get rid of the forcing and let β be very large (the system is overdamped). Now the equation reads

$$\dot{x} \quad -x + x^2$$

Sketch this curve and locate the fixed points. What does the potential look like? (Remember $f(x) = -dV(x)/dx$). Now we see what happens when capsize occurs. The ship "tips over the edge" of the potential and can't recover.

We're now interested in the conditions that will lead to capsize. Again, in an effort to simplify things let's ignore the forcing for a moment. So our equation reads

$$\ddot{x} + \dot{x}\beta + x - x^2 = 0$$

We transform this to two first order equations by defining a new variable $v = \dot{x}$ so that the equation becomes

$$\begin{aligned} v &= \dot{x} \\ \dot{v} &= -v\beta - x + x^2 \end{aligned} \qquad (2)$$

Now we have to numerically integrate this and see which initial conditions lead to capsize. The numerical integration can be done by extending the Runge-Kutta method to two-dimensions (see Strogatz page 33 for 1-D and page 147 for 2-D**). Before doing any computing you should first be able to sketch what the phase portrait looks like. Can you identify any fixed points and classify their stability? Using *Differential Systems* and a value of β=0.1 investigate the behavior of the system from a number of different starting points. Try to locate the bounds of stability, that is, the initial conditions which lead to capsize and those that do not.**

You have been supplied with a computer program in `Matlab` (**stable.m** which calls john_ode.m and unforced.m) that you can use to obtain the stable region, the set of initial conditions in $\dot{x}$ and x for which the ship will not capsize. This program calls a Runge-Kutta routine to perform the numerical integration. What we'll do is set the system off with a number of initial conditions and see which conditions lead to capsize and which result in the ship settling down. **Run the program.** A value of β=0.1 has been set for the damping. **The program has been preset with a time, t_f, for which the Runge-Kutta integration will take place. Is this**

value a good choice? What are the drawbacks of decreasing and increasing this time? Can you predict without running the program again what the effect of changing β will be? Now check the prediction by trying β= 0.01 and β=0.5. You can change the value of β by editing the program unforced.m. **Explain qualitatively what happens as β is increased.** The program stable.m makes an array called "basin" that holds a "zero" when the initial condition does not lead to capsize and a "one" when the ship does capsize. The set of initial conditions that do not lead to capsize is often referred to as the basin of attraction. You can get a printout of this array by typing "colormap(gray)" followed by "imagesc(basin)" and typing print. **Print out the image that shows the stable region and <u>put in the axes by hand.</u>**

<u>Putting in the forcing</u>

Let's now go back to our forced equation, but remove the nonlinearity. We now have an equation for a damped, driven and LINEAR oscillator.

$$\ddot{x} + \dot{x}\beta + x = F\sin(\omega t)$$

We'll examine what happens as a function of frequency and the amplitude of the forcing. This equation is very well known in physics and exhibits a phenomenon known as resonance. That is, at a specific forcing frequency the response of the system is maximum. You may remember from the spring oscillator and the LRC circuit that their equations are the same as the left hand side of the above equation and exhibit a natural frequency of oscillation. If one forces the oscillator at this frequency then the response is maximum.

Again use *Differential Systems* to explore the behavior. You will have to modify equation (2) in the computer. We run the numerical integration for a number of forcing periods so that the system has settled down after the initial transient. (You may want to start with around 20 periods, which you can do by setting the limit of the integration time to a suitable value by trial and error). The response is taken as the amplitude of the ship's roll at the end of the forcing. **Set the amplitude to F = 0.05, β=0.1, the initial $x = \dot{x} = 0$, and determine the amplitude of the oscillation as a function of frequency (for ω = 0.4 to 1.2 in steps of 0.1). Plot the response as a function of frequency (you can do this by hand). Record the frequency at which the response of the system is maximum. We call this the resonant frequency.**

Now we put back in the nonlinearity. What do you think its effect will be? **Repeat the simulation, using a forcing amplitude of F= 0.05. Where does resonance occur?** Now you see that the nonlinearity has the effect of "softening" the resonance (that is, making the resonance occur at a lower frequency). Note also that there is now a sharp discontinuity. The ship's amplitude jumps suddenly. In this particular case the jump is not fatal, but it shows that in the presence of nonlinearity the behavior of the system can change dramatically with a small change in the inputs.

At this stage of the simulation we could either elect to investigate stability as a function of inital conditions or as a function of forcing frequency and amplitude. For now we'll attempt to sketch the regimes of capsize and stability as a function of forcing amplitude and forcing frequency, keeping the initial $\dot{x}$ and x fixed (both at values of 0). This means we'll have transients where the ship capsizes before it settles down to its long-term behavior and depends on the initial conditions of the simulations. **For values of forcing frequency from ω=0.3 to 1.5 in steps of 0.1 vary**

the forcing amplitude. Observe the output and sketch a graph separating the regimes of stability and instability. The graph should look a little like figure 2.

One can now also start to see other strange behavior in this system. For example, as you scan in amplitude you'll see regions where the response of the system is no longer at the period of the forcing. In fact, the period is sometimes twice that of the forcing, a phenomenon called period doubling. You can find such a place at $\omega = 0.3$ and $F = 0.24$. **Check that the period is in fact doubled.** For other values of ω and F the response doesn't seem to have any periodicity and in fact seems random before capsize occurs. This is the regime of chaos, but note that a ship may capsize during the initial transient as well. It is very difficult to separate these behaviors in our current simulation. Try to shade in (roughly) the regions where period doubling and this random looking behaviour occurs. Are they near where the capsize occurs?

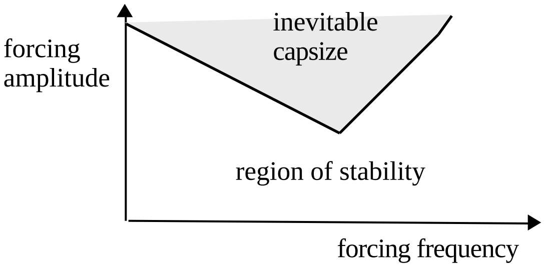

Figure 2: Schematic of regions of stability and instability.

Whats up in the phase space?

You may have been wondering what is happening in phase space. Unfortunately the phase space is now three-dimensional and is spanned by $(x, \dot{x}, t)$. The trajectories look like the wires of a complicated cable and wrap around each other. If we plot the phase space $(x, \dot{x})$, the plot looks tangled. This makes visualization quite difficult but a standard technique for dealing with this is to sample the $(x, \dot{x})$ whenever t is a multiple of $T = 2\pi/\omega$ so what we now see is a set of dots that appear to puncture a two-dimensional sheet - this is called the Poincare Map. We can use this Poincare map to visualize the behavior of three (and higher) dimensional systems and will come back to it later in the course.
Have a look using Differential Systems. Generate phase plots of $\dot{x}$ vs. x using an equation with no forcing. Now include a sin($\omega\tau$) term in the equation and examine the trajectories starting from the origin in phase space. You'll see the trajectories no longer have that "well groomed" look to them that Strogatz describes (page 149). Generate the phase plots when you are (a) far from resonance, (b) near to resonance and (c) when the system is period doubled. Describe the appearance of these plots and explain what happens when the system makes the transition to period doubling.

Lab Report:
In the lab report you should cover the points that are highlighted in the text. These points are reiterated below along with some other points that you should address.
1) Show the potential and stable points.
2) Find the basin of stability (the set of initial conditions for which the ship will not capsize) and explain what happens as parameters are varied.
3) Show where the resonances occur for linear and nonlinear oscillators.
4) What sort of potential would lead to a "hardening" of the resonance?

5) Prepare a graph showing the combinations of forcing frequency and amplitude for which the ship becomes unstable.
6) Prepare plots of the phase space for the forced and unforced equation. Describe what happens when the motion is period doubled.
7) In other ship stability studies a potential of the form

$$V = \frac{1}{2}x^2 - \frac{1}{4}x^4 + bx$$

has been used where b is a constant. Sketch this potential and the associated restoring force showing any fixed point(s).

References:

Soliman, M. S. and J. M. T. Thompson. "Transient and steady state analysis of capsize phenomena." *Applied Ocean Research*, **13**(2), 82-92, (1991)

Bishop, S. R. and M. S. Soliman, "The prediction of ship capsize: not all fractals are environment friendly." *Applications of Fractals and Chaos*. Crilly, Earnshaw and Jones (Eds). Springer-Verlag 1993