

MULTI-DIMENSIONAL SYSTEM: Coupled Oscillators

In this experiment we will explore a multi-dimensional system. In general, multi-dimensional systems are governed by equations of the form

$$\dot{x}_1 = f_1(x_1, x_2, \dots, x_n)$$

$$\dot{x}_2 = f_2(x_1, x_2, \dots, x_n)$$

.

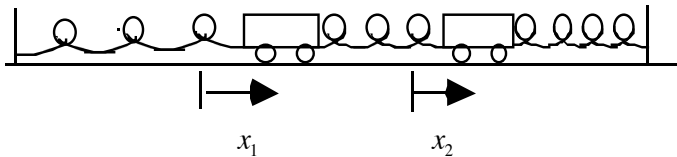
.

$$\dot{x}_n = f_n(x_1, x_2, \dots, x_n)$$

where n is the dimension of the system.

This system consists of two carts on a track, each connected to a spring on one side and coupled together by a spring, as shown in the figure.

In this system the two carts are identical with mass m , and all the springs have the same spring constant, k . Consider displacing cart 1 and cart 2 from their equilibrium positions by x_1 and x_2 respectively, as shown in the figure:



One can write the equation of motion for each cart by considering the force applied by each spring on each mass and including a damping term proportional to the velocity of each cart. The resulting equations are:

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) - b\dot{x}_1 = -2kx_1 + kx_2 - b\dot{x}_1$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) - b\dot{x}_2 = -2kx_2 + kx_1 - b\dot{x}_2$$

Eqns. 1

This system can be represented as a 4-dimensional system by defining the velocity of each mass as a dynamical variable. We replace the \ddot{x} terms with \dot{v} , where $v = \dot{x}$, and we obtain:

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2 - \frac{b}{m}v_1$$

$$\dot{v}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2 - \frac{b}{m}v_2$$

Eqns. 2

Do you see why this system is linear? Do you see why all 4 dynamical variables are coupled? As you have seen in chapter 5, linear systems of two or more coupled dimensions can

be analyzed by finding the eigenvalues and eigenvectors. The following approach is somewhat equivalent to this method, but is a more brute force technique.

If one defines new variables (these are the eigenvectors) as

$$\begin{aligned} y &= x_1 + x_2 \\ z &= x_1 - x_2 \\ v_y &= \dot{x}_1 + \dot{x}_2 \\ v_z &= \dot{x}_1 - \dot{x}_2 \end{aligned} \quad \text{Eqns. 3}$$

then Eqns. 2 will uncouple and turn into two decoupled sets of 2-dimensional systems, with each set describing a damped oscillator:

$$\begin{aligned} \dot{y} &= v_y \\ \dot{v}_y &= -\frac{k}{m}y - \frac{b}{m}v_y \\ \dot{z} &= v_z \\ \dot{v}_z &= -3\frac{k}{m}z - \frac{b}{m}v_z \end{aligned} \quad \text{Eqns. 4}$$

The two 2-dimensional systems are decoupled because the “y” variables only depend on each other and not on the “z” variables (the reverse is also true). Note that the first set describes an oscillator with natural angular frequency $\omega_y = \sqrt{k/m}$ and the second set describes an oscillator with $\omega_z = \sqrt{3k/m}$. These are called the normal modes of this coupled system. In one mode the variable y changes like a sine function and the frequency of the oscillations is $\sqrt{k/m}$. In the other mode the variable z changes like a sine function and the frequency of the oscillations is $\sqrt{3k/m}$.

One can start the system so that only one normal mode is observed. The y -normal mode can be obtained by displacing both of the carts the same distance in the same direction initially. The z -normal mode can be obtained by displacing the carts the same distance in opposite directions (either towards each other or away from each other) initially. Any arbitrary initial condition will give you a combination of these normal modes, and the system will have both of the frequencies present.

In this experiment we will observe the system in both normal modes and also choose an arbitrary initial condition and verify that the motion is a linear combination of both modes.

Actions and Questions:

We will measure the displacement of each cart by measuring the force on each cart. This is necessary because the interface we use only allows one motion detector, but can take data from two force probes simultaneously. Since the force is proportional to the displacement in a mass-spring system, and the spring constants are all equal, the force probes give a suitable measure of the displacement of each cart. The force probes are attached to the carts and the springs should be connected to the force probes on one side.

1- Open up the *Logger Pro* file called **coupled**.

2- Calibrate each force probe.

(a) To do this, disconnect the spring from the probe and open the calibration window via **Experiment/Calibrate**.

(b) Decide which force probe you want to calibrate and check which port that force probe is connected to (either DIN1 or DIN2). Choose this one from the list in the calibration window and click **Perform Now**.

(c) For your calibration points choose 0 N (**Reading 1**) and 2.5 N (**Reading 2**). Use the Newton spring scale for the 2.5 N calibration point.

(d) Repeat steps (b) and (c) for the other force probe.

3- Connect the carts to the springs and let the system come to equilibrium. Zero the force probes using **Experiment/Zero/Zero All Sensors**. Check that the signs for each force probe are correct and consistent with the positive x -direction that you choose. If you need to change it, do this using **Experiment/Calibrate** and the **Reverse Direction** button.

4- Set the system oscillating in the y -normal mode by displacing both carts in the same direction and by the same amount. Release the carts from rest and collect data. Do several trials until you get good clean data for x_1 and x_2 . Plot x_1 vs time and x_2 vs time on the same graph. Next plot (x_1+x_2) vs. time and (x_1-x_2) vs. time on the same graph. You will need to create new columns for this data. Do this by choosing *new column* under **Data** item on the menu and make a new column by adding/subtracting the data from each probe.

◆ Print, label and attach the plots to your lab report.

◆ Do the graphs confirm that the system is in the y -normal mode? How can you tell? Use your graphs to calculate the natural frequency ω ($=2\pi/\text{Period}$) for this mode.

Save this file to the desktop and then Data/Clear All Data.

5- Now set the system oscillating in the z -normal mode by pushing them in opposite directions and by the same amount. Release the carts from rest and collect data. Do several trials until you get good clean data for x_1 and x_2 .

◆ Repeat step 4 (note that your new columns and graphs should already be created).

Save this file to the desktop and then Data/Clear All Data.

6- According to Eqns. 4 the two normal mode frequencies are related by a factor of $\sqrt{3}$.

◆ Check the frequencies you obtained to see if this relationship holds.

7- Now start from some arbitrary initial condition (e.g. push only one cart away from its equilibrium and release) and collect data.

◆ Repeat step 4.

◆ Do the x_1 vs time and x_2 vs time graphs look sinusoidal (i.e. can each be described by a simple sine function)?

◆ Do the (x_1+x_2) vs. time and (x_1-x_2) vs. time graphs look sinusoidal (i.e. can each be described by a simple sine function)? If so, estimate the natural frequency ω for each one. Do these values make sense?

8- Let us now think about the phase portrait for this system. Since the system is 4-dimensional, the phase portrait should also be 4-dimensional. The phase portrait described by the coupled set of equations (Eqns 2) is quite complex. To get a sense of its complexity, use your data from step 7 to make a plot of \dot{x}_1 vs x_1 . You will need to create a new column for the \dot{x}_1 data (make sure you use the *smooth derivative* option).

◆ Print this phase portrait.

Note: You know from class that the trajectory in a phase portrait may not cross itself. It may appear that the trajectory in your phase plot crosses itself, but remember that your plot is just a 2-dimensional projection of the real 4 dimensional plot. Only the projection of the trajectory is crossing itself.

8- The phase portrait from step 7 is not particularly illuminating. It makes more sense to plot separate phase portraits for the two decoupled systems, i.e. \dot{y} vs y and \dot{z} vs z .

◆ Print these phase portraits. Do they make sense? How do they compare to the simple harmonic oscillator experiment that you did last week?

8- ◆ Sketch the phase portraits (\dot{y} vs y and \dot{z} vs z) that you would expect if you set the carts in motion by initially displacing both carts in the same direction, by the same amount.

◆ Sketch the phase portraits (\dot{y} vs y and \dot{z} vs z) that you would expect if you set the carts in motion by initially by pushing them in opposite directions the same amount.