

## TWO-DIMENSIONAL SYSTEMS: Simple Harmonic Oscillator-Computer

**Objective:** To introduce the use of the phase plane and numerical differential equation solvers for two-dimensional systems.

**Tools:** Computers with software such as *Differential Systems* or *MacMath*. The software should allow initial conditions to be set using a mouse.

**Topics:** the mass-spring system (simple harmonic oscillator)

We can model a variety of physical systems by means of second order differential equations, but we can solve only a few of them analytically. In this lab we will introduce the phase plane for a 2-D linear system by means of a differential equations solver. The featured system will be a mass and a spring, without and then with damping.

A two dimensional dynamical system that is independent of time can be specified by

- a phase portrait
- the vector field

We'll look at

- the graph of the solutions in the  $(t, x)$  –plane with position  $x$  vs. time  $t$
- the graph of trajectories in the  $(x, v)$  –plane (phase plane) with velocity  $v$  vs. position  $x$
- how both graphs are affected by varying the parameters in the differential equation
- the interpretation of vector field marks in the phase plane

The differential equation for the unforced mass-spring system with damping is

$$m \ddot{x} + b \dot{x} + k x = 0 \quad (1)$$

where  $m$  denotes the mass,  $b$  the damping constant, and  $k$  the spring constant. In order to graph the solutions to this equation we must rewrite it as a system of two 1-D equations. We let

$$\dot{x} = v$$

Then since  $\dot{v} = \ddot{x}$ , we need only solve for  $\ddot{x}$  in equation (1) and substitute to get

$$\dot{v} = \ddot{x}$$

$$\dot{v} = -\frac{b}{m} v - \frac{k}{m} x \quad (2)$$

### 1. The undamped mass spring

We'll use the undamped case to illustrate the process. If we set  $b = 0$  and let  $\omega^2 = \frac{k}{m}$ , the set of equations (2) becomes:

$$\dot{x} = v$$

$$\dot{v} = -\omega^2 x$$

**1.1** Then for initial values

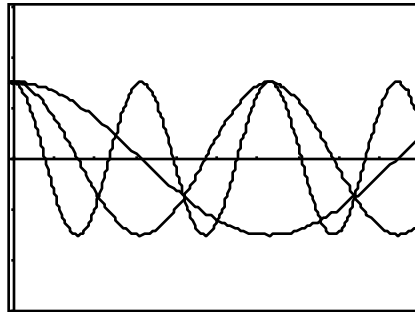
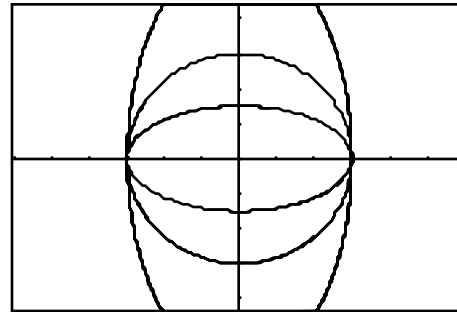
$$x(0) = 4$$

$$v(0) = 0$$

and  $\omega = 0.5, 1, 2$ ; plot all the solutions on the same  $x$  vs.  $t$  graph and all of the trajectories on the same  $v$  vs.  $x$  graph, so that they can be compared for different values of the parameter.

**Software note:** With *Differential Systems*, you can include the parameter  $\omega$  (or for convenience, use  $w$ ) in the equations. The software allows you to reset the value of  $\omega$  and display new trajectories superimposed on the same graph as shown below. Note that a ruler can be displayed for ease in reading the graphs.

Label each curve shown with the appropriate value of  $\omega$ .

View:  $x$  vs.  $t$ View:  $v$  vs.  $x$ 

**1.2** Refer to the graphs shown. As  $\omega$  increases, we can see that the frequency of the oscillations increase. In fact,  $\omega$  is called the **angular frequency** (in radians/second). What happens in the phase plane? How are  $x$  and  $v$  affected as  $\omega$  is increased? You can check your answer by plotting  $v$  vs.  $t$ .

## 2. The effect of damping

Let's simplify our system by assuming that  $m = 1$ ,  $k = 4$  for some convenient system of units. Then our system of equations becomes:

$$\dot{x} = v$$

$$\dot{v} = -bv - 4x$$

**2.1** For  $b = 0, 1, 2, 3, 4, 5$ , plot  $x$  vs.  $t$  and  $v$  vs.  $x$  for the initial conditions used in the earlier examples. Again put all of the  $x$  vs.  $t$  plots on one graph. And also put the  $v$  vs.  $x$  trajectories on one graph. Show a ruler and select your plot bounds carefully.

**Software note:** When using *Differential Systems*, use the method of writing the system of equations in terms of the parameter  $b$ . Then vary  $b$  by assigning values to  $b$  in the box supplied by the software.

**2.2** Using pen or pencil, label each trajectory on each graph with the appropriate value of  $b$ .

**2.3** It can be shown that critical damping occurs when  $b = \sqrt{4mk}$ . Does this seem consistent with your graphical results? If your results are rather ambiguous, you may need to change the scale of the graphing plane to see greater detail.

Indicate the appropriate trajectory on the phase plane and the  $x$  vs.  $t$  graph. You can use a "thick pen" option on the computer or use another color pencil or ink to highlight the trajectory for critical damping on each graph. Include these graphs in your report.

**2.4** Are your results consistent with the notions of underdamping and overdamping? Which trajectories represent underdamped motion?

**2.5** Look at the phase plane. As the value of  $b$  increases, does the trajectory spiral in toward the fixed point at the origin more quickly or less quickly? Discuss.

**2.6.** Now compare the trajectories in the phase plane for the undamped and damped mass-spring systems. Interpret the meaning of closed trajectories and spiraling trajectories in terms of periodicity. Explain what this property means in terms of conservative systems.

### 3. Vector Field Marks

Now consider the 2-D system

$$\dot{x} = v$$

$$\dot{v} = -0.1v - x$$

**3.1** We want to look at the field line through the point (1,-2) on the phase plane. Before we plot the field lines, calculate the slope.

$$\frac{dv}{dx} = \frac{dv/dt}{dx/dt} = \frac{\dot{v}}{\dot{x}}$$

So at (1,-2), where  $x = 1$  and  $v = -2$ , calculate

$$\frac{dv}{dx} =$$

**3.2** Plot the field marks, plot the trajectory through the point (1, -2) and print out the graph. (Note that the field marks are usually plotted with a uniform length. With *Differential Systems* you can add arrows to the field marks as well as changing the density and length.)

On the printed graph, plot your calculated slope at (1, -2). Is your field mark tangent to the trajectory at this point?

### Other kinds of 2-D systems

We've used the damped unforced oscillator to illustrate the ease of using a differential equations solver to analyze the behavior of a simple 2-D system providing we can model it with a second order differential equation.

Notice that we did not need to write down a solution. In fact for more complicated 2-D systems, closed form analytical solutions may not exist. Yet we can use the same software in the same fashion to learn about the system. We are limited only by our ability to construct a model consistent with observed behavior, the limitations of the numerical techniques used by the software, and our ability to interpret the results.

Some complications are:

- non-linearity
- explicit time dependence
- non-constant coefficients

Many of the properties we'll be looking at in the future, such as behavior of trajectories around fixed points, can be explored by looking at trajectories in the phase plane.