A SYMPLECTIC FIXED POINT THEOREM ON OPEN MANIFOLDS

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Abstract. In 1968 Bourgin proved that every measure-preserving, orientation-preserving homeomorphism of the open disk has a fixed point, and he asked whether such a result held in higher dimensions. Asimov, in 1976, constructed counterexamples in all higher dimensions. In this paper we answer a weakened form of Bourgin's question dealing with symplectic diffeomorphisms: every symplectic diffeomorphism of an even-dimensional cell sufficiently close to the identity in the $C^1$-fine topology has a fixed point. This result follows from a more general result on open manifolds and symplectic diffeomorphisms.

Introduction. Fixed point theorems for area-preserving mappings have a history which dates back to Poincaré's “last geometric theorem”, i.e., any area-preserving mapping of an annulus which twists the boundary curves in opposite directions has at least two fixed points. More recently it has been proved that any area-preserving, orientation-preserving mapping of the two-dimensional sphere into itself possesses at least two distinct fixed points (see [N, Si]). In the setting of noncompact manifolds, Bourgin [B] showed that any measure-preserving, orientation-preserving homeomorphism of the open two-cell $B^2$ has a fixed point. For Bourgin's theorem one assumes that the measure is finite on $B^2$ and that the measure of a nonempty open set is positive. Bourgin also gave a counterexample to the generalization of the theorem for the open ball in $R^3$ and asked the question whether his theorem remains valid for the open balls in low dimensions. In [As] Asimov constructed counterexamples for all dimensions greater than two and actually got a flow of measure-preserving, orientation-preserving diffeomorphisms with no periodic points.

To formulate our results and place the comments above into our framework, we need some concepts from symplectic geometry. A smooth manifold is called symplectic if there exists a nondegenerate, closed, differentiable 2-form $\omega$ defined on $M$. A differentiable mapping $f$ of $M$ into itself is called symplectic if $f$ preserves the form $\omega$. We refer to the texts by Abraham and Marsden [A & M] and Arnold [A] for the general background in symplectic geometry.

We reformulate Bourgin's question to ask: does every symplectic mapping of a $2n$-dimensional cell, equipped with a symplectic structure, have a fixed point? Using a generalization of a theorem of Weinstein [W], we answer this question affirmatively for mappings sufficiently close to the identity.
1. Preliminaries. All manifolds are assumed to be finite-dimensional, $C^\infty$-smooth, and without boundary. A manifold $M$ is open if $M$ has no compact components. Let $e(M)$ denote the ends of $M$, and let $\hat{M} = M \cup e(M)$ be the completion of $M$. We consider manifolds $M$ where the number of ends, denoted by $e(M)$, is finite and where $\hat{M}$ has a smooth manifold structure without boundary. For the general problem of completing an open manifold with finitely many ends see Siebenmann’s thesis [S].

If $M$ is a manifold with symplectic form $\omega$, then $\text{Diff}(M, \omega)$ denotes the group of symplectic diffeomorphisms of $M$. The closed one-forms on $M$ are denoted by $Z^1(M)$. Both of these function spaces are topologized with the $C^1$-fine topology. See [H, p. 35] for a good account of the $C^1$-fine topology.

We require the basic formalism of “cotangent co-ordinates” contained in the following theorem of Weinstein.

**Theorem 1.1** [W1, Proposition (2.7.4) or W2, Theorem 7.2]. If $(M, \omega)$ is a symplectic manifold, then there is a $C^1$-fine neighborhood $A \subset \text{Diff}(M, \omega)$ containing the identity map, a $C^1$-fine neighborhood $B \subset Z^1(M)$ containing the zero form, and a homeomorphism $V: A \to B$. If $f \in A$, then a point $x \in M$ is a fixed point of $f$ if and only if $(V(f))(x) = 0$.

**Proof.** If $f$ is in $\text{Diff}(M, \omega)$, then the graph of $f$ is a Lagrangian submanifold of $M \times M$ with the symplectic structure $\pi_1^*\omega - \pi_2^*\omega$, where $\pi_1$ and $\pi_2$ are the projections. There exists a neighborhood $U$ of the diagonal $\Delta(M) = \{(m, m): m \in M\}$ and a bijection of $U$ onto a neighborhood $W$ of the zero-section in $T^*M$, taking Lagrangian submanifolds of $U$ onto Lagrangian submanifolds lying in $W$. If $f$ is close enough to the identity, in the sense that the graph of $f$ is contained in $U$, then there is a one-form $V(f) \in Z^1(M)$ whose image is contained in $W$. Clearly, $f(x) = x$ if and only if $(V(f))(x) = 0$. □

Various fixed point theorems in symplectic geometry result from Theorem 1.1. For examples see [M, N, S, W1, and W2]. Let $M$ be a compact manifold and $\eta$ a closed one-form. Define $c(\eta)$ to be the number of zeros of $\eta$. Define $c(M) = \text{glb} \{c(\eta): \eta \in Z^1(M)\}$. If $M$ is a symplectic manifold with symplectic form $\omega$, then there is a $C^1$-neighborhood of $\text{id}_M$ in $\text{Diff}(M, \omega)$, so that if $f$ is in this neighborhood, then $V(f)$ is a closed one-form. Furthermore, the number of fixed points of $f$ is equal to $c(V(f))$. Now assume $M$ is simply connected, so that every closed one-form is exact. Then $c(M) \geq 2$ since every smooth function on a compact manifold has at least two critical points. Therefore, in this $C^1$-neighborhood of $\text{id}_M$ every $f$ has at least two fixed points.

2. The main theorem. When the manifold $M$ is not compact there are functions with no critical points, and hence there are closed one-forms with no zeros. Therefore, $c(M) = 0$. In this section we extend the fixed point theorem of Weinstein to open symplectic manifolds. Note that while $M$ may be a symplectic manifold, its completion $\hat{M}$ may carry no symplectic structure at all. In particular, for the open
2n-cell $B^{2n} = \{ x \in \mathbb{R}^{2n}: \| x \| < 1 \}$ the completion is homeomorphic to $S^{2n}$, which has no symplectic structure for $n > 1$. The open manifold $B^{2n}$ has the standard symplectic structure induced from $\mathbb{R}^{2n}$.

**Theorem 2.1.** If $(M, \omega)$ is a symplectic manifold with $e(M) < c(\tilde{M})$, then there exists a $C^1$-fine neighborhood $A$ of $\text{id}_M$ in $\text{Diff}(M, \omega)$ such that every $f \in A$ has at least $c(\tilde{M}) - e(M)$ fixed points.

**Proof.** Assume $M$ is embedded in $\tilde{M}$ as an open submanifold. Let $\phi: \tilde{M} \to \mathbb{R}$ be a nonnegative function vanishing only on the ends of $M$, $\phi(x) = 0$ if and only if $x \in \tilde{M} - M$. Let $B \subset Z^1(M)$ be the set of one-forms defined by $\phi$,

$$B = \{ \eta \in Z^1(M): \| \eta(x) \| < \phi(x), \| D\eta(x) \| < \phi(x) \}$$

where the norms arise from a riemannian metric on $\tilde{M}$. So $B$ is an open subset and every $\eta \in B$ extends to a form $\tilde{\eta}$ on $\tilde{M}$ such that $\tilde{\eta}(x) = 0$ for $x \in \tilde{M} - M$. By taking an intersection, if necessary, we may assume that $B$ satisfies the conclusions of Theorem 1.1. Since $c(\tilde{M}) - e(M) > 0$ and $c(\tilde{\eta}) \geq c(\tilde{M})$, it follows that $c(\tilde{\eta}) - e(M) > 0$, so that $\tilde{\eta}$ has more zeros than there are points in $\tilde{M} - M$. Therefore $\eta(x) = 0$ for some $x \in M$. Now we use Theorem 1.1 to get a $C^1$-fine neighborhood $A$ in $\text{Diff}(M, \omega)$ containing the identity and a homomorphism $V: A \to B$. For $f \in A$, the one-form $V(f)$ is in $B$ and so $f$ has a fixed point $x$ in $M$. $\Box$

We now restrict our attention to manifolds $M$ diffeomorphic to $\mathbb{R}^{2n}$. Let $\omega$ be any symplectic structure on $M$. Clearly, $e(M) = 1$ and by picking a point $N \in S^{2n}$, we can embed $M$ onto $S^{2n} - \{N\}$, so that $\tilde{M} \approx S^{2n}$. With this construction and the fact that $c(S^{2n}) = 2$, we have

**Corollary 2.2.** Let $(M, \omega)$ be a symplectic manifold where $M$ is diffeomorphic to $\mathbb{R}^{2n}$. Then there is a neighborhood in the $C^1$-fine topology of $\text{Diff}(M, \omega)$ which contains $\text{id}_M$, such that every mapping in this neighborhood has a fixed point.

One should be aware that there are symplectic diffeomorphisms of $\mathbb{R}^{2n}$ with symplectic structure $\Sigma dx_i \wedge dy_i$ that have no fixed points, in particular the translations, but there are $C^1$-fine neighborhoods of the identity containing no translations. Let $\phi: \mathbb{R}^{2n} \to \mathbb{R}^+$ be a function vanishing at infinity and use $\phi$ to define an open neighborhood consisting of the diffeomorphisms $f$ such that $\| f(x) - x \| < \phi(x)$ and $\| Df(x) - I \| < \phi(x)$ for all $x \in \mathbb{R}^{2n}$.

**References**


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