

SINGULAR FACTORS ARE RARE

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ABSTRACT. We prove that for H^p functions $f(z)$ and $g(z)$ which have mutually prime singular factors, $f(z) - wg(z)$ has a trivial singular inner factor for all complex numbers w , except at most for those w in a set of logarithmic capacity zero.

Let ϕ be an analytic function in the Hardy space H^p on the open unit disc $\Delta = \{z : |z| < 1\}$ for $0 < p \leq \infty$. It is classical that ϕ has a factorization $\phi = BSF$ where B is a Blaschke product, S is a singular function, and F is an outer function. Specifically, these factors are

$$B(z) = z^m \prod \mu_k \frac{z_k - z}{1 - z\bar{z}_k}, \quad \mu_k = \frac{|z_k|}{z_k}$$

where m is the order of the zero of ϕ at the origin and z_1, z_2, \dots are the zeros of ϕ in $\Delta \setminus \{0\}$;

$$S(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\nu(t) \right\}$$

where ν is a non-negative measure singular with respect to Lebesgue measure; and

$$F(z) = \lambda \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |\phi(e^{it})| dt \right\}$$

where λ is a unimodular constant. See [2] for a full description of these functions and their properties.

It is a well-known theorem of Frostman that if ϕ is an inner function, then $\frac{w - \phi(z)}{1 - \bar{w}\phi(z)}$ is a Blaschke product for all $w \in \Delta$ with the exception at most of a set of capacity zero.

Caughran and Shields [1] raised this question: how big is the set of complex numbers w such that $\phi(z) - w$ has a non-trivial singular function as a factor? They showed that if ϕ' is in the Hardy space H^1 then the set of such w is countable. Fisher [3] showed, with no assumption on ϕ' , that the set of w for which the singular function has an atom in its associated measure is countable.

A theorem of Rudin [8, Theorem 4] can be used to give an answer to the above question. The theorem deals with functions of n complex variables, but we will use the restriction to $n = 1$. The class N_* is defined (in [8]) to be the set of all analytic functions f on the unit disc such that the functions $\log^+ |f_r|$ have uniformly absolutely continuous integrals. (Here f_r is defined by $f_r(z) = f(rz)$, $z \in \Delta$ and $r \in (0, 1)$.) What this means explicitly is that to each $\epsilon > 0$ there should exist a

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$\delta > 0$ such that

$$\int_A \log^+ |f(rw)| dm(w) < \epsilon$$

for all $A \subset T$ (the unit circle) with $m(A) < \delta$, and for all $r \in (0, 1)$. The class N is the usual Nevanlinna class, which can be viewed as the space of all functions on the unit disc which are quotients of bounded analytic functions. N^+ is the class of all functions on the unit disc which can be written as the quotient of a bounded analytic function with an outer function. See [2] for details about these classes. It is left as an exercise for the reader to see that $N_* \subset N$, and, in fact, $N_* = N^+$. The following is then a corollary to Rudin's theorem:

Theorem 1. *Let $\phi \in N_*$. Then the set of points w for which $\phi(z) - w$ has a non-trivial singular inner factor has logarithmic capacity zero. Conversely, given any (compact) set E of logarithmic capacity zero, there is a bounded analytic function ϕ such that $\phi(z) - w$ has a non-trivial singular inner factor if and only if $w \in E$.*

The converse statement is well-known; see [3]. Let E be a compact set of capacity zero in Δ . The covering map F of the domain $\Delta \setminus E$ is an inner function since E has capacity zero. For each $w \in E$, $\frac{F(z)-w}{1-\bar{w}F(z)}$ is a non-vanishing inner function and so is singular. Thus, since $1 - \bar{w}F(z)$ is an outer function, $F(z) - w$ is a function with nontrivial singular inner factor for all w in E .

Sarason produced a different sort of extension of Frostman's result, which appears in a paper by R. Mortini [7] as part of a constructive proof of the Beurling–Rudin theorem. He proved that for mutually prime inner functions u and v (by which we mean that u and v have no zero in common and that there is no singular inner function S with $u = Su_1$ and $v = Sv_1$ for inner functions u_1 and v_1), and $\rho > 0$, the function $u(z) + \rho e^{it}v(z)$ has a trivial singular inner factor for almost all (with respect to Lebesgue measure) real t .

Here we provide a generalization to the theorems of Frostman, Rudin, and Sarason to further answer the general question of when singular inner factors disappear.

Theorem 2. *Let $f, g \in H^p$, $0 < p \leq \infty$, have mutually prime singular inner factors. Then the set of points w for which $f(z) - wg(z)$ has a non-trivial singular inner factor has logarithmic capacity zero.*

Remark. In our Theorem 2, if g is an outer function, then we see that the lack of a singular factor in $f(z) - wg(z)$ is equivalent to the lack of a singular factor in the decomposition of the function $\frac{f(z)}{g(z)} - w$ (in N_*), and is thus covered in Theorem 1.

Proof. We write

$$(1) \quad f(z) - wg(z) = F_w(z)B_w(z)S_w(z)$$

where the product on the right-hand side of (1) is of the outer, Blaschke product, and singular inner factors, respectively. We first comment on the dependence of the factors on w . Since

$$|F_w(z)| = \exp \left\{ \frac{1}{2\pi} \int P_z(e^{i\theta}) \log |f(e^{i\theta}) - wg(e^{i\theta})| d\theta \right\}$$

where P_z is the Poisson kernel for $z \in \Delta$, it follows that $|F_w(z)|$ is a continuous function of w on Δ . That is, if $w_n \rightarrow \zeta$ and we write $f(z) - w_n g(z) = F_n(z)B_n(z)S_n(z)$

and $f(z) - \zeta g(z) = F(z)B(z)S(z)$, then $|F_n| \rightarrow |F|$ uniformly on compact subsets of Δ . Hence, $|B_n S_n| \rightarrow |BS|$ uniformly on compact subsets of Δ .

Suppose now that E is a compact set of positive capacity. By adjusting by a scale factor, we may assume that $\max_E |w| < \frac{1}{2}$. There is a positive measure μ on E with integral 1, such that

$$(2) \quad v(z) = \int_E \log |w - z| d\mu(w)$$

is continuous in the whole plane [6, Theorem 1.8 and Ch. I, §3]. Since μ has compact support, $v(z)$ is also bounded below.

The function u defined by

$$u(z) = \int_E \log |B_w(z)S_w(z)| d\mu(w)$$

is non-positive in Δ . Moreover,

$$(3) \quad u(z) = \int_E \log |f(z) - wg(z)| d\mu(w) - \int_E \log |F_w(z)| d\mu(w).$$

Let $d\sigma$ be the normalized Lebesgue measure on the unit circle T . We shall show that

$$(4) \quad \lim_{r \rightarrow 1} \int_T u(re^{i\theta}) d\sigma(\theta) = 0.$$

This will give us the first equation in the following:

$$\begin{aligned} 0 &= \lim_{r \rightarrow 1} \int_T \int_E \log |B_w(re^{i\theta})S_w(re^{i\theta})| d\mu(w) d\sigma(\theta) \\ &= \lim_{r \rightarrow 1} \int_E \left\{ \int_T \log |B_w(re^{i\theta})S_w(re^{i\theta})| d\sigma(\theta) \right\} d\mu(w) \\ &\leq \int_E \left\{ \limsup_{r \rightarrow 1} \int_T \log |B_w(re^{i\theta})S_w(re^{i\theta})| d\sigma(\theta) \right\} d\mu(w). \end{aligned}$$

This last inequality holds by Fatou's lemma, since the term inside the bracket is nonpositive. Hence

$$\lim_{r \rightarrow 1} \int_T \log |B_w(re^{i\theta})S_w(re^{i\theta})| d\sigma(\theta) = 0, \quad \text{a.e. } [\mu],$$

and thus $B_w(z)S_w(z)$ is a Blaschke product [5, problem 6, page 75]. Thus, $S_w(z)$ is constant for w outside a set of μ -measure zero.

We proceed to prove (4) by considering two terms:

$$\begin{aligned} I(r) &= \int \int \log |f(re^{i\theta}) - wg(re^{i\theta})| d\mu(w) d\sigma(\theta) \quad \text{and} \\ II(r) &= \int \int \log |F_w(re^{i\theta})| d\mu(w) d\sigma(\theta). \end{aligned}$$

We wish to show that, for

$$G(z) = \int_E \log |f(z) - wg(z)| d\mu(w)$$

we have

$$(5) \quad \begin{aligned} \lim_{r \rightarrow 1} I(r) &= \lim_{r \rightarrow 1} \int G(re^{i\theta}) d\sigma(\theta) \\ &= \int G(e^{i\theta}) d\sigma(\theta) = \iint \log |f(e^{i\theta}) - wg(e^{i\theta})| d\mu(w) d\sigma(\theta). \end{aligned}$$

Note here that

$$\begin{aligned} G(z) &= \int_E \log |g(z)| d\mu(w) + \int_E \log \left| \frac{f(z)}{g(z)} - w \right| d\mu(w) \\ &= \log |g(z)| + v \left(\frac{f(z)}{g(z)} \right) \end{aligned}$$

and that μ was chosen so that $v(z)$ is continuous, so we easily see that for almost every θ ,

$$\lim_{r \rightarrow 1} G(re^{i\theta}) = G(e^{i\theta}).$$

Equation (5) will then be true by a variant of the dominated convergence theorem provided there is a family of non-negative integrable functions $V_r(\theta)$ and an integrable function V such that $V_r(\theta) \rightarrow V(\theta)$,

$$\lim_{r \rightarrow 1} \int V_r(\theta) d\sigma(\theta) = \int V(\theta) d\sigma,$$

and

$$|G(re^{i\theta})| \leq V_r(\theta).$$

To demonstrate the existence of such a family $V_r(\theta)$, we will show separately that $G(re^{i\theta})$ has such an upper bound and a lower bound. We will need to use the factorization of f and g into outer and inner factors: $f(z) = O_f(z)I_f(z)$ and $g(z) = O_g(z)I_g(z)$.

To find the upper bound, note that for every $w \in E$,

$$(6) \quad |f(re^{i\theta}) - wg(re^{i\theta})| \leq |O_f(re^{i\theta})| + |O_g(re^{i\theta})|.$$

This leads us to choose

$$(7) \quad \begin{aligned} V_r(\theta) &= \log^+ |O_f(re^{i\theta})| + \log^+ |O_g(re^{i\theta})| + 1 \quad \text{and} \\ V(\theta) &= \log^+ |O_f(e^{i\theta})| + \log^+ |O_g(e^{i\theta})| + 1. \end{aligned}$$

We can easily see by combining (6) and (7) (and using the inequality $\log(a+b) \leq \log^+ a + \log^+ b + 1$, for all real $a, b > 0$) that

$$\begin{aligned} G(re^{i\theta}) &= \int_E \log |f(re^{i\theta}) - wg(re^{i\theta})| d\mu(w) \\ &\leq \log(|O_f(re^{i\theta})| + |O_g(re^{i\theta})|) \\ &\leq V_r(\theta) \end{aligned}$$

and it is clear that $V_r(\theta) \rightarrow V(\theta)$ pointwise. Since O_f and O_g are outer functions, we get $\int V_r(\theta) d\sigma \rightarrow \int V(\theta) d\sigma$ [2, Theorem 2.10].

To find the lower bound, we need the following lemma:

Lemma 1. *There is a constant K such that for all $z \in \Delta$,*

$$(8) \quad G(z) \geq \log(\max\{|f(z)|, |g(z)|\}) + K.$$

For the proof of the lemma, we break the unit disc up into two pieces, A and B , where A consists of those points z where $|g(z)| \geq |f(z)|$, and B those points where $|g(z)| < |f(z)|$. We will prove the lemma separately for points in A and points in B .

If $z \in A$, then $\log(\max\{|f(z)|, |g(z)|\}) = \log |g(z)|$. Recall that

$$G(z) = \log |g(z)| + v \left(\frac{f(z)}{g(z)} \right)$$

and $v(z)$, defined in equation (2), is bounded below, so the term on the right above can be written as in (8).

If $z \in B$, then we note that $|f(z) - wg(z)| > \frac{1}{2}|f(z)|$, so

$$G(z) = \int_E \log |f(z) - wg(z)| d\mu(w) > \log\left(\frac{1}{2}|f(z)|\right).$$

Also, for $z \in B$, $\log(\max\{|f(z)|, |g(z)|\}) = \log |f(z)|$, and again, (8) can be satisfied. This completes the proof of the lemma.

Now we will find the family $V_r(\theta)$ just as before. We note that

$$(9) \quad \log(\max\{|f(re^{i\theta})|, |g(re^{i\theta})|\}) \geq \log^- |O_f(re^{i\theta})| + \log^- |O_g(re^{i\theta})| \\ + \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\}).$$

Take

$$(10) \quad V_r(\theta) = \log^- |O_f(re^{i\theta})| + \log^- |O_g(re^{i\theta})| + \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\})$$

and

$$V(\theta) = \log^- |O_f(e^{i\theta})| + \log^- |O_g(e^{i\theta})|$$

Putting together (8), (9), and (10) gives us

$$G(re^{i\theta}) \geq V_r(\theta) + K.$$

The sum of the first two terms in the definition of $V_r(\theta)$ in (10) clearly approaches $V(\theta)$ pointwise, and the third term approaches zero pointwise. Furthermore,

$$\int_T V_r(\theta) d\sigma = \int_T \log^- |O_f(re^{i\theta})| + \log^- |O_g(re^{i\theta})| d\sigma(\theta) \\ + \int_T \log(\max\{|I_f(re^{i\theta})|, |I_g(re^{i\theta})|\}) d\sigma(\theta),$$

and the first term on the right, just as in the upper bound case, approaches $\int V(\theta) d\sigma(\theta)$. The second term approaches zero, which we can see from the following lemma by Sarason:

Lemma 2. *If u_1 and u_2 are inner functions without a common factor, then*

$$\lim_{r \rightarrow 1} \int_T \log(\max\{|u_1(re^{i\theta})|, |u_2(re^{i\theta})|\}) d\sigma(\theta) = 0.$$

Sarason's proof of this lemma can be found in [7], but we include it here, with his permission, for completeness. The limit on the left side is the value at the origin of the least harmonic majorant in Δ of the subharmonic function $\max\{\log |u_1|, \log |u_2|\}$. So it remains to show that this least harmonic majorant is the constant function 0. Let h denote this least harmonic majorant. Then $\log |u_1| \leq h \leq 0$. This implies that h has radial limits 0 almost everywhere on T .

So, if h is not identically zero, then h is the Poisson integral of a negative singular measure on T . Hence $\varphi = e^{h+i\tilde{h}}$ is a singular inner function (here \tilde{h} denotes the harmonic conjugate of h in Δ). Since $|u_1| \leq |e^{h+i\tilde{h}}|$, the inner function φ divides u_1 . But $|u_2| \leq |e^{h+i\tilde{h}}|$ implies that φ also divides u_2 , contradicting our assumption about u_1 and u_2 . Thus $h \equiv 0$. This completes the proof of the lemma.

Our conditions for convergence of the integrals in (5) are satisfied, so we get

$$(11) \quad \lim_{r \rightarrow 1} I(r) = \iint \log |f(e^{i\theta}) - wg(e^{i\theta})| d\sigma(\theta) d\mu(w).$$

Next,

$$\begin{aligned} II(r) &= \iint \log |F_w(re^{i\theta})| d\mu(w) d\sigma(\theta) \\ &= \iiint \log |f(e^{it}) - wg(e^{it})| P_r(\theta - t) d\sigma(t) d\sigma(\theta) d\mu(w) \\ (12) \quad &= \iint \log |f(e^{it}) - wg(e^{it})| d\sigma(t) d\mu(w). \end{aligned}$$

We have $\int u(re^{i\theta}) d\sigma(\theta) = I(r) - II(r)$. When (11) and (12) are used in this, we find $\lim_{r \rightarrow 1} \int u(re^{i\theta}) d\sigma(\theta) = 0$. As explained earlier, this finishes the proof that the set of w for which $f(z) - wg(z)$ has a nontrivial singular factor has capacity zero. \square

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