

# THE ESSENTIAL NORM OF A COMPOSITION OPERATOR ON A PLANAR DOMAIN

STEPHEN D. FISHER AND JONATHAN E. SHAPIRO

ABSTRACT. We generalize to finitely connected planar domains the result of Joel Shapiro which gives a formula for the essential norm of a composition operator. In the process, we define and give some properties of a generalization of the Nevanlinna counting function and prove generalizations of the Littlewood inequality, the Littlewood–Paley identity, and change of variable formulas, as well.

## 1. INTRODUCTION

Let  $\Omega$  be a domain in the plane. For  $1 \leq p < \infty$ , the Hardy space  $H^p = H^p(\Omega)$  is defined to be those analytic functions  $f$  on  $\Omega$  for which the subharmonic function  $|f(z)|^p$  has a harmonic majorant. Once we specify a base point  $t_0 \in \Omega$ , we define the norm of  $f$  to be  $p^{\text{th}}$  root of the value at  $t_0$  of the (unique) least harmonic majorant of  $|f|^p$ . A different choice of the base point gives an equivalent norm on  $H^p$ ; this is an application of Harnack's inequality. The Hardy space  $H^\infty$  is the space of bounded analytic functions on  $\Omega$  with the supremum norm. For more on the Hardy spaces, see [6], [1].

An analytic function  $\varphi$  that maps  $\Omega$  into itself determines a composition operator  $C_\varphi$  on  $H^p$  given by

$$(1) \quad C_\varphi f = f \circ \varphi.$$

$C_\varphi$  is a bounded operator on  $H^p$ . One simple way to see this is to note that if  $u_f$  is the least harmonic majorant of  $|f|^p$ , then  $u_f \circ \varphi$  is an harmonic majorant of  $|f \circ \varphi|^p$  and so

$$\|f \circ \varphi\|^p \leq u_f(\varphi(t_0)) \leq K u_f(t_0)$$

where  $K$  is a constant that, again by Harnack's inequality, depends only on the domain  $\Omega$ , and the points  $t_0$  and  $\varphi(t_0)$ .

In this paper we are concerned with  $H^p$  on a domain  $\Omega$  that is finitely-connected; that is, has only a finite number of complementary components. In this setting, it is known [2] that  $C_\varphi$  is compact on some  $H^p$ ,  $1 \leq p < \infty$ , if

---

*Date:* August 29, 1997.

1991 *Mathematics Subject Classification.* Primary 47B38, Secondary 30H05, 46E20.

and only if it is compact on all  $H^p$ . We therefore concentrate on  $C_\varphi$  acting on  $H^2$ . The main result of this paper is an extension of the theorem of Joel Shapiro [7] on the essential norm — distance to the set of compact operators — of the composition operator  $C_\varphi$  that he proved when  $\Omega$  is the unit disk. To understand the statement of Shapiro's theorem we must first define the Nevanlinna counting function of  $\varphi$ .

**Definition 1.1.** *Let  $\Delta$  be the open unit disk and suppose that  $\varphi$  is an analytic function mapping  $\Delta$  into itself. The Nevanlinna counting function for  $\varphi$  is, for  $w \neq \varphi(0)$ ,*

$$(2) \quad N_\varphi(w) = \sum_{\varphi(z)=w} -\log |z|.$$

With this background, we can state Joel Shapiro's result.

**Theorem 1.2.** *Suppose that  $\varphi$  is an analytic function that maps  $\Delta$  into  $\Delta$  with  $\varphi(0) = 0$ . Let  $\|C_\varphi\|_e$  denote the essential norm of  $C_\varphi$  as an operator on  $H^2$ . Then*

$$\|C_\varphi\|_e^2 = \limsup_{|w| \rightarrow 1^-} \left[ \frac{N_\varphi(w)}{-\log |w|} \right].$$

*In particular,  $C_\varphi$  is compact on  $H^2$  if and only if*

$$\lim_{|w| \rightarrow 1^-} \frac{N_\varphi(w)}{-\log |w|} = 0.$$

The development of this paper follows the arguments of Shapiro in [7] closely, altering several parts as necessary to allow for the change in setting.

## 2. BACKGROUND

Let  $D$  be a domain in the plane whose universal covering surface is the open unit disc  $\Delta$  and let  $\Pi$  be the covering map. The *Poincaré metric* for  $D$  is defined at  $\zeta = \Pi(z) \in D$  by

$$\lambda_D(\zeta) = |\Pi'(z)|(1 - |z|^2).$$

It is shown in [3, p. 44] that the value of  $\lambda_D(\zeta)$  is independent of the particular choice of  $z \in \Delta$  with  $\Pi(z) = \zeta$ .

If  $D$  is regular for the Dirichlet problem, we denote the Green's function for  $D$  with pole at  $p \in D$  by  $g_D(z; p)$ . The domain  $D$  is omitted unless confusion is possible.

In this paper we shall generally be concerned with a planar domain  $\Omega$  whose complement consists of a finite number of disjoint non-trivial continua. Such a domain is conformally equivalent to one whose boundary consists of a finite number of disjoint analytic simple closed curves; indeed, it is conformally equivalent to a domain whose boundary components are

circles. Since the conformal mapping gives an isometry of the corresponding Hardy spaces, we may assume, and shall do so, that the components  $\Gamma_0, \dots, \Gamma_p$  of  $\Gamma$  are circles, with  $\Gamma_0$  the boundary of the unbounded component of the complement of  $\Omega$ . We let  $\omega_{t_0}$  denote the harmonic measure on  $\Gamma$  for the (fixed) base point  $t_0$ . It is standard [2] that each  $H^2$  function  $f$  on  $\Omega$  has boundary values almost everywhere on  $\Gamma$ , that these boundary values lie in  $L^2(\Gamma, \omega_{t_0})$ , and that the correspondence of  $f$  to its boundary values is an isometry of  $H^2$  onto a closed subspace of  $L^2(\Gamma, \omega_{t_0})$ . We let  $\Omega_j$  be the region outside  $\Gamma_j, j = 1, \dots, p$ , including the point at  $\infty$  and  $\Omega_0$  be the region inside  $\Gamma_0$ . Each of the regions  $\Omega_j$  is conformally equivalent to the unit disk  $\Delta$  via a linear fractional transformation. When we write  $H^2(\Omega_j)$  for the Hardy space for this region, we shall always assume that the norm is taken with respect to the base point  $t_0$ .

**2.1. Factorization of  $H^p$  functions.** There is a factorization of functions in  $H^p(\Omega)$ , developed in [8], that parallels that for  $H^p$  functions on the unit disc. Here we give a summary; additional details may be found in [1, Section 4.7].

Let  $\mathcal{G}$  be the group of linear fractional transformations of  $\Delta$  onto itself that leave the covering map  $\Pi$  invariant:  $\Pi \circ \tau = \Pi, \tau \in \mathcal{G}$ . An analytic function  $h$  on  $\Delta$  is *modulus automorphic* if for each  $\tau \in \mathcal{G}$  there is a unimodular constant  $c = c(\tau)$  such that  $h \circ \tau = ch$ . Each modulus automorphic function  $h$  corresponds to a function  $f$  on  $\Omega$  by  $h(z) = f(\Pi(z)), z \in \Delta$ . The modulus of  $f$  is single-valued, but  $f$  itself has unimodular periods in the sense that analytic continuation of a function element  $(f, \mathcal{O})$  along any curve  $\gamma$  in  $\Omega$  leads to the function element  $(cf, \mathcal{O})$ , where  $c$  is a unimodular constant that depends only on the homotopy class of  $\gamma$ . The class of such multiple-valued analytic functions with single-valued modulus whose  $p^{\text{th}}$  power has a harmonic majorant will be denoted by  $MH^p(\Omega)$ .

A *Blaschke product*  $B$  is an element of  $MH^\infty(\Omega)$  with

$$\log |B(z)| = - \sum_k g_\Omega(z; w_k), \quad \sum_k g_\Omega(w_k; t_0) < \infty.$$

If there are only a finite number of zeros, then the second condition is automatically satisfied.

A *singular inner function*  $S$  is an element of  $MH^\infty$  with

$$\log |S(z)| = - \int_\Gamma P(s; z) d\nu(s)$$

where  $\nu$  is a non-negative Borel measure on  $\Gamma$  that is singular with respect to harmonic measure  $\omega_{t_0}$  and  $P(\cdot; z)$  is the Poisson kernel for  $z \in \Omega$ .

An *outer function* in  $MH^p$  is an element  $F$  of  $MH^p$  of the form

$$\log |F(z)| = \int_\Gamma u(s) P(s; z) d\omega_{t_0}(s)$$

where  $u \in L^1(\Gamma, \omega_{t_0})$  and  $e^u \in L^p(\Gamma, \omega_{t_0})$ .

The basic theorem on factorization is this.

**Theorem 2.1.** *Each function  $f \in MH^p(\Omega)$  has a factorization as*

$$f = BSF$$

where  $B$  is a Blaschke product,  $S$  is a singular inner function, and  $F$  is an outer function in  $MH^p(\Omega)$ . The factors are unique up to multiplication by unimodular constant. Even if  $f$  is single-valued, the factors need not be.

**2.2. The Nevanlinna counting function.** Our first goal is to generalize the Nevanlinna counting function to the domain  $\Omega$  and understand some of its properties.

**Definition 2.2.** *Let  $\varphi : \Omega \rightarrow \Omega$  be an analytic function. The Nevanlinna counting function for  $\varphi$ ,  $N_\varphi(w)$  for  $w \in \Omega \setminus \{\varphi(t_0)\}$ , is*

$$N_\varphi(w) = \sum_{\varphi(z)=w} g_\Omega(z; t_0).$$

Note that this reduces to the counting function defined previously if  $\Omega$  is the unit disk  $\Delta$  and  $t_0 = 0$ .

For the Nevanlinna counting function on the unit disk, there is the classical theorem of Littlewood [4]:

**Theorem 2.3.** *Let  $\psi$  be a holomorphic self-map of the unit disk  $\Delta$ . Then*

$$(3) \quad N_\psi(w) \leq \log \left| \frac{1 - \overline{\varphi(0)}w}{\varphi(0) - w} \right|, \quad w \in \Delta \setminus \{\psi(0)\}$$

with equality holding for quasi-every  $w$  (i.e., all  $w$  except those in a set of capacity zero) exactly when  $\psi$  is inner.

If  $\psi(0) = 0$ , then (3) reduces to

$$N_\psi(w) \leq -\log |w|$$

which is an improvement of the Schwarz inequality.

For counting functions on  $\Omega$ , we have the following generalization of Littlewood's inequality:

**Theorem 2.4.** *Let  $\varphi : \Omega \rightarrow \Omega$  be analytic and fix the point  $t_0$ . Then*

$$N_\varphi(w) = \sum_{\varphi(z)=w} g(z; t_0) \leq g(w; t_0) \text{ for all } w \in \Omega \setminus \{t_0\},$$

with equality holding (for quasi-every  $w$ ) exactly when  $\varphi(\Gamma) \subset \Gamma$ , by which we will mean that the boundary values of  $\varphi$  on  $\Gamma$  lie in  $\Gamma$  almost everywhere (with respect to  $\omega_{t_0}$ ).

*Proof:* Let  $g(z; w)$  be the Green's function for  $\Omega$  with pole at  $w$ . The function  $g(\varphi(z); w)$  is harmonic on  $\Omega$  except at the collection of isolated points where  $\varphi(z) = w$ ; at such a point  $g(\varphi(z); w)$  has a logarithmic pole. Let  ${}^*g(\varphi(z); w)$  be the (multiple-valued) harmonic conjugate of  $g(\varphi(z); w)$  on  $\Omega \setminus \{\varphi(z) = w\}$  and set

$$Q_w(z) = e^{-g(\varphi(z); w) - i{}^*g(\varphi(z); w)}.$$

$Q_w$  lies in  $MH^\infty$ ; indeed, its modulus is bounded by one. Using Theorem 2.1, we factor  $Q_w$  in  $H^2(\Omega)$  as  $Q_w(z) = B_w(z)S_w(z)F_w(z)$  where the factors are a Blaschke product, a singular inner function, and an outer function, respectively. We then get

$$\begin{aligned} -\log |Q_w(t_0)| &= g(\varphi(t_0); w) \\ &= g(t_0; w) \\ &= -\log |B_w(t_0)| - \log |S_w(t_0)| - \log |F_w(t_0)|. \end{aligned}$$

The function  $Q_w(z)$  has zeros exactly where  $\varphi(z) = w$ , so we have

$$-\log |B_w(t_0)| = \sum_{\varphi(z)=w} g(t_0; z) = N_\varphi(w).$$

Thus we see that

$$g(w; t_0) = N_\varphi(w) - \log |S_w(t_0)| - \log |F_w(t_0)|,$$

so

$$g(w; t_0) \geq N_\varphi(w).$$

We have equality when both  $\log |S_w(t_0)|$  and  $\log |F_w(t_0)|$  are zero, which happens when both  $S_w$  and  $F_w$  are unimodular constants. If  $\varphi(\Gamma) \subset \Gamma$ , then we will have  $|Q_w| = 1$  almost everywhere on  $\Gamma$  and thus  $F_w \equiv 1$ . By the extension of Frostman's theorem which is proved below, Theorem 2.6, since  $Q_w$  is a Blaschke product composed with  $\varphi$ , it has trivial singular factor for quasi-every  $w$  in  $\Omega$ .  $\square$

The well-known theorem of Frostman, for functions on the unit disk, can be stated

**Theorem 2.5.** *Let  $\psi$  be an inner function on  $\Delta$ . Then for  $|w| < 1$ , the function*

$$(4) \quad q_w(z) = \frac{\psi(z) - w}{1 - \overline{w}\psi(z)}$$

*is a Blaschke product except possibly for a set of  $w$  in  $\Delta$  of logarithmic capacity zero.*

For our generalization, we will prove the following theorem and associated lemma, which are suggested in [1, Ch. 5, Exercise 2,3]

**Theorem 2.6.** *Let  $\varphi$  be an analytic function on  $\Omega$  with  $\varphi(\Gamma) \subset \Gamma$ , and  $B_w(z) = \exp\{-g(z; w) - i^*g(z; w)\}$  be the Blaschke product on  $\Omega$  with zero at  $w$ . Then the function*

$$Q_w(z) = B_w(\varphi(z))$$

*is a Blaschke product (on  $\Omega$ ), except possibly for a set of  $w$  in  $\Omega$  of logarithmic capacity zero.*

*Proof:* For  $\Pi$  the universal covering map from  $\Delta$  onto  $\Omega$  (with  $\Pi(0) = t_0$ ), we have the pull-back map  $\psi : \Delta \rightarrow \Delta$  which satisfies  $\varphi \circ \Pi = \Pi \circ \psi$ . It is easy to see that if  $\varphi(\Gamma) \subset \Gamma$ , then  $\psi$  must be inner. Define

$$E = \left\{ w \in \Delta : \frac{\psi(z) - w}{1 - \bar{w}\psi(z)} \text{ has a nontrivial singular factor} \right\}.$$

By Theorem 2.5 above,  $E$  has logarithmic capacity zero, thus so does  $\Pi(E)$  (in  $\Omega$ ). We write

$$\begin{aligned} Q_w \circ \Pi &= B_w \circ \varphi \circ \Pi \\ &= B_w \circ \Pi \circ \psi. \end{aligned}$$

By Lemma 2.7, below,  $B_w \circ \Pi$  is a Blaschke product on  $\Delta$ , with zeros at those points  $z$  with  $\Pi(z) = w$ . The function  $B_w \circ \Pi \circ \psi$  is thus a (constant times a) product of terms of the form (4). If  $w$  is not in  $\Pi(E)$ , then each of these terms is a Blaschke product. Thus  $B_w \circ \Pi \circ \psi = B_w \circ \varphi \circ \Pi$  is a Blaschke product, and, again by Lemma 2.7,  $B_w \circ \varphi = Q_w$  is a Blaschke product on  $\Omega$ .  $\square$

**Lemma 2.7.** *The analytic function  $B$  on  $\Omega$  is a Blaschke product if and only if  $B \circ \Pi$  is a Blaschke product on  $\Delta$ .*

*Proof:* If  $B$  is a Blaschke product on  $\Omega$ , we can write

$$B(z) = e^{-\sum g(z; z_j) - i^*(\sum g(z; z_j))}$$

for some sequence  $\{z_j\}$  with the property that  $\sum_1^\infty g(\zeta; z_j) < \infty$  for each  $\zeta \in \Omega$ .  $B \circ \Pi$  is easily seen to be an inner function on  $\Delta$ , so we can write

$$(5) \quad B \circ \Pi = bS,$$

where  $b$  is a Blaschke product on  $\Delta$  and  $S$  is a singular inner function. The Blaschke product  $b$  has a zero at any  $z$  with  $\Pi(z) = z_j$  for some  $j$ . We now see that

$$\begin{aligned} -\log |B \circ \Pi(0)| &= -\log |B(t_0)| \\ &= \sum_j g(t_0; z_j) \\ &= \sum_j \sum_{\Pi(z)=z_j} \log \frac{1}{|z|} \\ &= -\log |b(0)|. \end{aligned}$$

The third line above comes from the fact [5, VII.5] that we can write the Green's function for  $\Omega$  in terms of Green's functions on the unit disk,

$$g(w; t_0) = \sum_{\Pi(a)=w} \log \frac{1}{|a|}.$$

But (5) gives us  $-\log |B \circ \Pi(0)| = -\log |b(0)| - \log |S(0)|$ , so  $|S(0)| = 1$ , and thus  $S \equiv 1$ , i.e.,  $B \circ \Pi = b$  is a Blaschke product in  $\Delta$ .

Now assume  $B \circ \Pi$  is a Blaschke product on  $\Delta$ . It is easy to see that  $B$  must be an inner function on  $\Omega$ , so it has the factorization in  $H^\infty(\Omega)$ ,

$$B = bS$$

where  $b$  is a Blaschke product on  $\Omega$  and  $S$  is a singular inner function on  $\Omega$  (i.e.,  $S$  has boundary values of modulus 1 a.e., and has no zeros on  $\Omega$ ). We then have

$$B \circ \Pi = (b \circ \Pi)(S \circ \Pi),$$

and we can easily see that  $S \circ \Pi$  is a function on  $\Delta$  which has no zeros and has boundary values of 1 a.e., so  $S \circ \Pi$  is a singular inner function. But  $B \circ \Pi$  is a Blaschke product, so has only trivial singular inner factor, i.e.,  $S \circ \Pi$  is trivial, so  $S$  must be trivial, and  $B$  must be a Blaschke product.  $\square$

**2.3. The sub-mean-value property.** We will need the following property for the counting function on  $\Omega$ :

**Theorem 2.8.** *Let  $h$  be an analytic function on a domain  $U$ . Suppose that  $D$  is an open disk in  $U \setminus h^{-1}(t_0)$  with center at  $a$  and that  $h(D) \subset \Omega$ . Then*

$$(6) \quad N_\varphi(h(a)) \leq \frac{1}{A(D)} \int_D N_\varphi(h(w)) dA(w)$$

where  $N_\varphi$  is the counting function for  $\Omega$  and  $A$  is area measure.

*Proof:* This sub-mean-value property follows from the version proved in [7], since we can express our counting function on  $\Omega$  as a counting function on the unit disk:

$$N_\varphi(w) = \sum_{\varphi(z)=w} g(z; t_0).$$

As we did earlier, we write the Green's function of  $\Omega$  in terms of the Green's function on  $\Delta$  to get

$$\begin{aligned} N_\varphi(w) &= \sum_{\varphi(z)=w} g(z; t_0) \\ &= \sum_{\varphi(z)=w} \sum_{\Pi(a)=z} \log \frac{1}{|a|} \\ &= \sum_{\varphi \circ \Pi(a)=w} \log \frac{1}{|a|} \\ &= N_{\varphi \circ \Pi}(w), \end{aligned}$$

for  $\Pi$  the universal covering map of the unit disk onto  $\Omega$  which maps 0 to  $t_0$ . In this last line,  $N_{\varphi \circ \Pi}(w)$  is the counting function on the unit disk. It is shown in [7] that  $N_{\varphi \circ \Pi}(h(w))$  has the required sub-mean-value property, so  $N_\varphi(h(w))$  has the same property.  $\square$

**2.4. The Littlewood–Paley identity.** For functions in  $H^2(\Delta)$ , we have the Littlewood–Paley identity [7]:

**Theorem 2.9.** *For functions  $f \in H^2(\Delta)$ ,*

$$\|f\|_{H^2(\Delta)}^2 = \frac{1}{2\pi} \int_T |f(e^{i\theta})|^2 d\theta = \frac{2}{\pi} \int_\Delta |f'(z)|^2 \log(1/|z|) dA(z) + |f(0)|^2.$$

The corresponding theorem on  $\Omega$  is

**Theorem 2.10.** *For functions  $f \in H^2(\Omega)$ ,*

$$\|f\|_{H^2(\Omega)}^2 = \int_\Gamma |f|^2 d\omega_{t_0} = \frac{2}{\pi} \int_\Omega |f'(z)|^2 g(z; t_0) dA + |f(t_0)|^2,$$

where  $\omega_{t_0}$  is harmonic measure on  $\Gamma$  for  $t_0$ .

*Proof:* Let  $r$  be a small positive number and let  $\Omega_r = \Omega \setminus \{z : |z - t_0| \leq r\}$ . The boundary of  $\Omega_r$  is  $\Gamma_r = \Gamma \cup \{z : |z - t_0| = r\}$ . We begin with Green's formula:

$$\int_{\Gamma_r} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \int_{\Omega_r} (u \Delta v - v \Delta u) dA.$$

We take  $u = |f|^2$  and  $v = g(\cdot; t_0)$ . We have

$$d\omega_{t_0} = \frac{-1}{2\pi} \frac{\partial v}{\partial n} ds \quad \text{and} \quad \Delta u = 4|f'|^2.$$

On the circle  $|z - t_0| = r$ , the normal derivative of  $v$  is the radial derivative and equals  $1/r$  plus a bounded term. This gives a term on the left-hand side of  $2\pi|f(t_0)|^2$  as  $r \rightarrow 0$ . On the other hand,  $v$  itself is  $\log r$  plus a bounded term and so the other term from the left-hand side goes to zero as  $r \rightarrow 0$ . On the right-hand side, the Laplacian of  $v$  is identically zero on  $\Omega_r$  and  $v$  itself is  $\log s$ ,  $0 < s \leq r$  plus a bounded term on the circle  $|z - t_0| = s$ . Thus, the right-hand side approaches  $-4 \int_\Omega |f'(z)|^2 dA$  as  $r \rightarrow 0$ . Rearrangement gives the conclusion.  $\square$

**2.5. Change of variable formulas.** We also have the change of variable formula for the disk, see [7],

**Theorem 2.11.** *For any positive, measurable function  $F$  on  $\Delta$ , and analytic self-map  $\psi$  of  $\Delta$ ,*

$$\int_\Delta F(\psi(z)) |\psi'(z)|^2 \log(1/|z|) dA(z) = \int_\Delta F N_\psi dA.$$

We will need the corresponding theorem on the domain  $\Omega$ .

**Theorem 2.12.** *For any positive, measurable function  $F$  on  $\Omega$ , and analytic self-map  $\varphi$  of  $\Omega$ ,*

$$\int_{\Omega} F(\varphi(z)) |\varphi'(z)|^2 g(z; t_0) dA(z) = \int_{\Omega} F(z) N_{\varphi}(z) dA(z).$$

*Proof:* The proof follows closely the one in [7]. Since  $\varphi$  is a local homeomorphism on the open set  $\Omega'$  formed by deleting from  $\Omega$  the zeros of  $\varphi'$ , there exists a countable collection  $\{R_j\}$  of disjoint open regions in  $\Omega'$  the union of whose closures is  $\Omega$ , and such that  $\varphi$  is one-to-one on each  $R_j$ . Let  $\psi_j$  denote the inverse of the restriction of  $\varphi$  to  $R_j$ , so that  $\psi_j$  is a one-to-one map taking  $\varphi(R_j)$  back onto  $R_j$ . By the usual change of variable formula applied on  $R_j$ , with  $z = \psi_j(w)$ :

$$\int_{R_j} F(\varphi(z)) |\varphi'(z)|^2 g(z; t_0) dA = \int_{\varphi(R_j)} F(w) g(\psi_j(w); t_0) dA(w).$$

Thus, if  $\chi_j$  denotes the characteristic function of the set  $\varphi(R_j)$ :

$$\int_{\Omega} (F \circ \varphi) |\varphi'| g(\cdot; t_0) dA = \int_{\Omega} F(w) \left\{ \sum_j \chi_j(w) g(\psi_j(w); t_0) \right\} dA(w).$$

This is the desired formula, since the term in curly braces on the right side of the equation above is  $N_{\varphi}(w)$ .  $\square$

We will also need the following version of this change of variable formula.

**Corollary 2.13.** *For each  $f$  analytic on  $\Omega$ ,*

$$\|f \circ \varphi\|_{H^2(\Omega)}^2 = \frac{2}{\pi} \int_{\Omega} |f'|^2 N_{\varphi} dA + |f(\varphi(t_0))|^2.$$

*Proof:* The generalized form of the Littlewood–Paley identity applied to  $f \circ \varphi$  yields:

$$\begin{aligned} \|f \circ \varphi\|_{H^2(\Omega)}^2 &= \frac{2}{\pi} \int |(f \circ \varphi)'(z)|^2 g(z; t_0) dA + |f(\varphi(t_0))|^2 \\ &= \frac{2}{\pi} \int |f' \circ \varphi|^2 |\varphi'|^2 g(z; t_0) dA + |f(\varphi(t_0))|^2 \end{aligned}$$

(by the chain rule). An application of the change of variable formula, with  $F = |f'|^2$ , completes the proof.  $\square$

## 2.6. A basis for $H^2(\Omega)$ .

**Theorem 2.14.** *Let  $\Omega$  be bounded by  $p+1$  disjoint circles,  $\Gamma_0, \dots, \Gamma_p$ , and let  $\Omega_i$ ,  $i = 0, 1, \dots, p$  be defined as at the beginning of this section. There is an orthonormal basis of  $H^2(\Omega)$ , say  $u_0, u_1, u_2, \dots$ , with this property: if  $f \in H^2$  has the expansion  $\sum_0^{\infty} c_k u_k$ , then  $f - \sum_0^{m(p+1)} c_k u_k$  has a zero at  $t_0$  of order at least  $m$ ,  $m = 1, 2, 3, \dots$*

*Proof:* Let  $\phi_j$  be the linear fractional transformation that maps  $\Omega_j$  onto the unit disk  $\Delta$ , normalized so that  $\phi_j(t_0) = 0$ . Let  $u_{jk} = \phi_j^k$ ; then  $\{u_{jk}\}_{k=0}^\infty$  is an orthonormal basis of  $H^2(\Omega_j)$ ,  $j = 0, \dots, p$  and  $u_{jk}$  vanishes to order  $k$  at  $t_0$ . Arrange the functions  $u_{jk}$  as

$$u_{00}, u_{10}, \dots, u_{p0}, u_{01}, u_{11}, \dots, u_{p1}, u_{02}, \dots$$

and renumber them as  $v_0, v_1, v_2, \dots$ . Now let  $E_n$  be the closed linear span of  $\{v_{n+1}, v_{n+2}, \dots\}$ ,  $n = 0, 1, 2, \dots$ . Finally, let  $u_m$  be the (normalized) projection of  $v_m$  onto the orthogonal complement in  $H^2$  of  $E_n$ .<sup>1</sup> We now check that these functions have the desired properties. Evidently, by its very construction,  $u_0$  is orthogonal to  $v_1, v_2, \dots$ . We write  $u_1 = v_1 + h_1$  where  $h_1 \in E_1$ . Hence,

$$0 = \langle u_0, v_1 \rangle = \langle u_0, u_1 \rangle - \langle u_0, h_1 \rangle = \langle u_0, u_1 \rangle .$$

and so  $u_0$  is orthogonal to  $u_1$ . Next,  $u_2$  is orthogonal to  $v_3, v_4, \dots$ . We write  $u_2 = v_2 + h_2$  where  $h_2 \in E_2$ . Hence,

$$0 = \langle u_0, v_2 \rangle = \langle u_0, u_2 \rangle - \langle u_0, h_2 \rangle = \langle u_0, u_2 \rangle$$

and

$$0 = \langle u_1, v_2 \rangle = \langle u_1, u_2 \rangle - \langle u_1, h_2 \rangle = \langle u_1, u_2 \rangle$$

so that  $u_2$  is orthogonal to both of  $u_0$  and  $u_1$ . In a similar way we can see that the functions  $u_0, u_1, u_2, \dots$  are mutually orthogonal. Next, it is easy to establish that the linear span of  $u_0, \dots, u_n$  is the orthogonal complement of  $E_n$ ,  $n = 0, 1, 2, \dots$  and so if  $v \in H^2$  is orthogonal to  $u_0, u_1, \dots$ , then

$$v \in \bigcap_{n=0}^{\infty} E_n.$$

and thus  $v = 0$ . Finally, suppose  $f \in H^2$  has an orthonormal expansion  $f = \sum c_k u_k$ . Those  $u_k$  with  $k > (p+1)m$  vanish at  $t_0$  to order at least  $m$  and hence  $f - \sum_0^{m(p+1)} c_k u_k$  has a zero at  $t_0$  of order at least  $m$ .

**Proposition 2.15.** *Suppose that  $f \in H^2(\Omega)$  vanishes to order  $n$  at  $t_0$ . Let  $\Pi$  be the universal covering map from  $\Delta$  to  $\Omega$ . For  $\zeta \in \Omega$  and  $\Pi(z) = \zeta$ , let  $\lambda_\Omega$  be the Poincaré metric for  $\Omega$ . Then*

- (a)  $|f(\zeta)| \leq \frac{|z|^n}{\sqrt{1-|z|^2}} (\lambda_\Omega(\zeta) \|\Pi\|_\infty)^{\frac{1}{2}} \|f\|_2$
- (b)  $|f'(\zeta)| \leq \sqrt{2n} \frac{|z|^{n-1}}{\sqrt{1-|z|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}}$ .

*Proof:* Let  $g = f \circ \Pi$  so that  $g$  has a zero of order  $n$  at the origin. Then use the standard estimates (see [7]) in the unit disk plus the fact that  $\lambda_\Omega(\zeta) = |\Pi'(z)|(1-|z|^2) \leq \|\Pi\|_{Bloch} \leq \|\Pi\|_\infty$ .

<sup>1</sup>Thanks to Todd Young for his contribution of this idea to the proof.

## 3. THE MAIN THEOREM

With the background of Section 2 in place, we are now ready to state and prove the main result of this paper.

**Theorem 3.1.** *Suppose that  $\Omega$  is finitely connected and that  $\varphi$  is an analytic function mapping  $\Omega$  into itself with  $\varphi(t_0) = t_0$ . Let  $\|C_\varphi\|_e$  denote the essential norm of  $C_\varphi$ , regarded as an operator on  $H^2(\Omega)$ . Then*

$$\|C_\varphi\|_e^2 = \limsup_{w \rightarrow \Gamma} \frac{N_\varphi(w)}{g(w; t_0)}.$$

*In particular,  $C_\varphi$  is compact on  $H^2$  if and only if*

$$\lim_{w \rightarrow \Gamma} \frac{N_\varphi(w)}{g(w; t_0)} = 0.$$

We will prove the theorem by proving separately upper and lower bounds for the essential norm of  $C_\varphi$ .

**3.1. The Upper Bound.** We will use the following general formula from [7] for the essential norm of a linear operator on a Hilbert space:

**Theorem 3.2.** *Suppose  $T$  is a bounded linear operator on a Hilbert space  $H$ . Let  $\{K_n\}$  be a sequence of compact self-adjoint operators on  $H$ , and write  $R_n = I - K_n$ . Suppose  $\|R_n\| = 1$  for each  $n$ , and  $\|R_n x\| \rightarrow 0$  for each  $x \in H$ . Then  $\|T\|_e = \lim_n \|TR_n\|$ .*

The goal now is to show that, for an analytic function  $\varphi : \Omega \rightarrow \Omega$  which fixes the point  $t_0$ ,

$$(7) \quad \|C_\varphi\|_e^2 \leq \limsup_{w \rightarrow \Gamma} \frac{N_\varphi(w)}{g(w; t_0)}.$$

We do this by applying the Proposition 3.2 above with  $K_n$  the operator which takes  $f$  to the sum of the first  $(p+1)n$  terms in its expansion relative to the basis we have chosen for  $H^2(\Omega)$  in Theorem 2.14.

For this orthonormal basis  $u_0, u_1, u_2, \dots$  of  $H^2(\Omega)$ , we can write any  $f \in H^2(\Omega)$  as  $f = \sum_{k=0}^{\infty} c_k u_k$ , and then  $K_n f = \sum_{k=0}^{(p+1)n} c_k u_k$ .  $R_n = I - K_n$  will then be an operator with the property that  $R_n f = \sum_{k=1+(p+1)n}^{\infty} c_k u_k$  has a zero of order at least  $n$  at  $t_0$ .

The operator  $K_n$  is self-adjoint and compact. Since  $R_n = I - K_n$ , its norm is 1, so that the hypotheses of the proposition are fulfilled, and

$$\|C_\varphi\|_e = \lim_{n \rightarrow \infty} \|C_\varphi R_n\|.$$

To estimate the right side of the above, fix a function  $f$  in the unit ball of  $H^2(\Omega)$ , and a positive integer  $n$ . Then by Corollary 2.13 we get

$$\|C_\varphi R_n f\|_{H^2(\Omega)}^2 = \frac{2}{\pi} \int |(R_n f)'|^2 N_\varphi dA + |R_n f(\varphi(t_0))|^2.$$

Since  $\|f\|_{H^2(\Omega)} \leq 1$ , the same is true of  $R_n f$ .

Now fix  $r < 1$ . Split the integral above into two parts,  $\Omega_r = \Pi(r\Delta)$  (where  $\Pi$  is the universal covering map of  $\Omega$  which maps the origin to  $t_0$ ), and the other its complement in  $\Omega$ ,  $\Omega_r^c$ . Then take the supremum of both sides of the resulting inequality over all functions  $f$  in the unit ball  $B$  of  $H^2(\Omega)$ . We obtain

$$\begin{aligned} \|C_\varphi R_n\|^2 &\leq \sup_B \frac{2}{\pi} \int_{\Omega_r} |(R_n f)'|^2 N_\varphi dA \\ &\quad + \sup_B \frac{2}{\pi} \int_{\Omega_r^c} |(R_n f)'|^2 N_\varphi dA + |R_n f(\varphi(t_0))|^2. \end{aligned}$$

We now use the pointwise estimate for  $(R_n f)'$ , from Proposition 2.15 part (b).  $R_n f$  has a zero of order at least  $n$  at  $t_0$ , and, for  $w \in \Omega_r$ ,  $w = \pi(z)$  for some  $z$  with  $|z| < r$ . Thus we have

$$|(R_n f)'(w)| \leq \sqrt{2n} \frac{|r|^{n-1}}{\sqrt{1-|r|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}},$$

so

$$\begin{aligned} \frac{2}{\pi} \int_{\Omega_r} |(R_n f)'|^2 N_\varphi dA \\ \leq \left( \sqrt{2n} \frac{|r|^{n-1}}{\sqrt{1-|r|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}} \right)^2 \frac{2}{\pi} \int_{\Omega_r} N_\varphi dA. \end{aligned}$$

Since the right side is  $(n|r|^{n-1})^2$  multiplied by terms which are bounded (independent of  $n$ ), we get

$$\frac{2}{\pi} \int_{\Omega_r} |(R_n f)'|^2 N_\varphi dA \rightarrow 0$$

as  $n \rightarrow \infty$ .

We can also easily see that  $|R_n f(\varphi(t_0))|^2 = |R_n f(t_0)|^2 = 0$  for all  $n \geq 1$ . As  $f$  runs through the unit ball of  $H^2(\Omega)$ ,  $R_n f$  runs through a subset of the ball, so we can replace  $R_n f$  in the remaining integral with  $f$  and only increase the right side. We use  $h(w) = N_\varphi(w)/g(w; t_0)$ , to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|C_\varphi R_n\|^2 &\leq \lim_{n \rightarrow \infty} \sup_B \frac{2}{\pi} \int_{\Omega_r^c} |(R_n f)'|^2 N_\varphi dA \\ &\leq \sup_B \frac{2}{\pi} \int_{\Omega_r^c} |f'(w)|^2 \frac{N_\varphi(w)}{g(w; t_0)} g(w; t_0) dA \\ &= \sup_B \frac{2}{\pi} \int_{\Omega_r^c} |f'(w)|^2 h(w) g(w; t_0) dA \\ &\leq \sup\{h(w) : w \in \Omega_r^c\} \sup_B \frac{2}{\pi} \int_\Omega |f'(w)|^2 g(w; t_0) dA \\ &\leq \sup\{h(w) : w \in \Omega_r^c\}, \end{aligned}$$

where the last line follows from the generalized version of the Littlewood–Paley identity, Theorem 2.10. As  $r \rightarrow 1$ ,  $w \in \Omega_r^c$  means that  $w$  is the image

under  $\Pi$  of only those  $z$  with  $|z| \geq r$ , so  $w \rightarrow \Gamma$ , thus

$$\sup\{h(w) : w \in \Omega_r^c\} \rightarrow \limsup_{w \rightarrow \Gamma} h(w),$$

giving us (7).

**3.2. The Lower Bound.** We now wish to complete the proof of Theorem 3.1 by showing

$$(8) \quad \|C_\varphi\|_e^2 \geq \limsup_{w \rightarrow \Gamma} \frac{N_\varphi(w)}{g(w; t_0)}.$$

The following elementary proposition will be used.

**Proposition 3.3.** *Suppose  $T$  is a bounded operator on a Banach space  $X$  and  $\{x_n\}$  is a sequence in the unit ball of  $X$  that goes weakly to zero. Then  $\|T\|_e \geq \limsup_{n \rightarrow \infty} \|Tx_n\|$ .*

*Proof:* Let  $K$  be a compact operator. Then

$$\|T - K\| \geq \limsup \|(T - K)x_n\| = \limsup \|Tx_n\|$$

since  $\|Kx_n\| \rightarrow 0$ . Now take the infimum over all  $K$  to get the desired conclusion.  $\square$

We will apply Proposition 3.3 to the operator  $C_\varphi$  with the role of  $\{x_n\}$  played by normalized reproducing kernels for the spaces  $H^2(\Omega_j)$ . We fix  $j, 0 \leq j \leq p$  and let  $\phi$  be the linear fractional transformation that maps  $\Omega_j$  onto the unit disk  $\Delta$ , with  $\phi(t_0) = 0$ . We then know that

$$\int_{\Gamma_j} u \circ \phi \, d\omega_j = \frac{1}{2\pi} \int_T u \, d\theta, \quad u \text{ continuous on } T$$

where  $\omega_j$  is the harmonic measure on  $\Gamma_j$  relative to  $\Omega_j$  for the point  $t_0$ . It then follows that the reproducing kernel for  $a$  on  $H^2(\Omega_j)$  is given by

$$K_a^{\Omega_j}(z) = \frac{1}{1 - \overline{\phi(a)}\phi(z)}.$$

From Proposition 3.3, we see that

$$(9) \quad \|C_\varphi\|_e^2 \geq \limsup_{a \rightarrow \Gamma_j} \frac{\|C_\varphi K_a^{\Omega_j}\|^2}{\|K_a^{\Omega_j}\|^2}.$$

We may compute terms on the right above in the following way:

$$\begin{aligned}
\|C_\varphi K_a^{\Omega_j}\|^2 &= \int_\Gamma |K_a^{\Omega_j} \circ \varphi|^2 d\omega_{t_0} \\
&= \frac{2}{\pi} \int_\Omega |K_a^{\Omega_j}(\varphi(z))| |\varphi'(z)|^2 dA(z) + |K_a^{\Omega_j}(t_0)|^2 \\
&= \frac{2}{\pi} \int_\Omega \frac{|\phi(a)|^2}{|1 - \overline{\phi(a)}\phi(\varphi(z))|^4} |\phi'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) + 1 \\
&= \frac{2}{\pi} \int_\Omega \frac{|\phi(a)|^2}{|1 - \overline{\phi(a)}\phi(w)|^4} |\phi'(w)|^2 N_\varphi(w) dA(w) + 1 \\
&= \frac{2}{\pi} \int_U \frac{|b|^2}{|1 - \overline{b}\zeta|^4} N_\varphi(\phi^{-1}(\zeta)) dA(\zeta) + 1,
\end{aligned}$$

where  $b = \phi(a)$ , and  $U = \phi(\Omega)$ . We now make the change of variables  $\zeta = \tau_b(\xi) = \frac{b-\xi}{1-\overline{b}\xi}$  or  $\xi = \tau_b(\zeta)$ . We then obtain

$$(10) \quad \|C_\varphi K_a^{\Omega_j}\|^2 \geq \frac{|b|^2}{(1-|b|^2)^2} \frac{2}{\pi} \int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi)$$

where  $U_b = \tau_b(U)$ .

We need to compute the norm of  $K_a^{\Omega_j}$  in  $H^2(\Omega)$  exactly. For simplicity, we set  $b = \phi(a)$ . Then

$$\begin{aligned}
\|K_a^{\Omega_j}\|^2 &= \int_\Gamma \left| \frac{1}{1 - \overline{b}\phi(z)} \right|^2 d\omega_{t_0}(z) \\
&= 1 + \frac{2}{\pi} \int_\Omega \frac{1}{|1 - \overline{b}\phi(z)|^4} |b|^2 |\phi'(z)|^2 g_\Omega(z; t_0) dA(z) \\
&= 1 + \frac{2}{\pi} \int_U \frac{1}{|1 - \overline{b}\zeta|^4} |b|^2 g_U(\zeta; 0) dA(\zeta) \quad (\text{where } U = \phi(\Omega)) \\
&= 1 + \frac{2}{\pi} \frac{|b|^2}{(1-|b|^2)^4} \int_{U_b} g_{U_b}(\xi; b) dA(\xi) \quad (\text{using } \xi = \frac{b-\zeta}{1-\overline{b}\zeta}).
\end{aligned}$$

This last integral may be computed using Theorem 2.10 with  $f(z) = z$ . This will give

$$\begin{aligned}
\frac{2}{\pi} \int_{U_b} g_{U_b}(\xi; b) dA(\xi) &= -|b|^2 + \int_{\partial U_b} |z|^2 d\omega_b^{U_b}(z) \\
&= -|b|^2 + \int_{\partial U} \left[ \frac{|\zeta - b|}{|1 - \overline{b}\zeta|} \right]^2 d\omega_0^U(\zeta) \\
&= -|b|^2 + 1 - (1 - |b|^2) \int_S \frac{1 - |z|^2}{|1 - \overline{b}z|^2} d\omega_0^U(z) \\
&= (1 - |b|^2) \left( 1 - \int_S \frac{1 - |z|^2}{|1 - \overline{b}z|^2} d\omega_0^U(z) \right)
\end{aligned}$$

where  $S$  is the union of that part of the boundary of  $U$  that lies inside  $\Delta$ , so that  $S$  is  $p$  disjoint circles lying inside  $\Delta$ . When this is substituted into the expression for  $\|K_a^{\Omega_j}\|^2$  we obtain

$$(11) \quad \|K_a^{\Omega_j}\|^2 = \left( \frac{1}{1-|b|^2} \right) \left( 1 - |b|^2 \int_S \frac{1-|z|^2}{|1-\bar{b}z|^2} d\omega_0^U \right)$$

$$(12) \quad = \left( \frac{1}{1-|b|^2} \right) (1 - I(b))$$

where

$$I(b) = |b|^2 \int_S \frac{1-|z|^2}{|1-\bar{b}z|^2} d\omega_0^U.$$

We now put (12) together with (10) and obtain

$$(13) \quad \frac{\|C_\varphi K_a^{\Omega_j}\|^2}{\|K_a^{\Omega_j}\|^2} \geq \frac{1}{(1-|b|^2)(1-I(b))} \frac{2}{\pi} \int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi).$$

The open set  $U = \phi(\Omega)$  has the form  $U = \Delta \setminus P$  where  $P$  is the union of  $p$  closed disks. As  $|b| \rightarrow 1$ ,  $\tau_b$  converges uniformly on compact subsets on  $\Delta$  to a unimodular constant. Thus,  $\tau_b(P)$  (as a subset of  $\Delta$ ) converges to the unit circle as  $|b| \rightarrow 1$ ; in particular, if  $r < 1$  is given, then  $U_b$  contains the disk  $\{|\xi| \leq r\}$  when  $|b|$  is near enough to 1; that is, when  $a$  is near enough to  $\Gamma_j$ . Thus,

$$\int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi) \geq \int_{|\xi| \leq r} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi).$$

However, by the sub-mean-value property for the counting function, Theorem 2.8, we obtain

$$\frac{2}{\pi} \int_{|\xi| \leq r} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi) \geq 2N_\varphi(\phi^{-1}(\tau(0)))r^2 = 2N_\varphi(a)r^2.$$

When this and (13) are applied to (9) we obtain

$$\|C_\varphi\|_e \geq \limsup_{a \rightarrow \Gamma_j} \frac{2}{(1-|b|^2)(1-I(b))} N_\varphi(a)r^2.$$

The number  $r$  may be arbitrarily near 1 so that it may be removed from this last inequality yielding

$$(14) \quad \|C_\varphi\|_e \geq \limsup_{a \rightarrow \Gamma_j} \frac{2N_\varphi(a)}{(1-|b|^2)(1-I(b))}, \quad b = \phi(a).$$

Next  $g_\Omega(a; t_0) = g_U(b; 0)$  and so

$$(15) \quad \frac{2}{(1-|b|^2)} N_\varphi(a) = \frac{N_\varphi(a)}{g_\Omega(a; t_0)} \frac{g_U(b; 0)}{\frac{1}{2}(1-|b|^2)}$$

We now claim that

$$(16) \quad \lim_{b \rightarrow e^{it}} \frac{g_U(b; 0)}{\frac{1}{2}(1 - |b|^2)(1 - I(b))} = 1, \quad 0 \leq t \leq 2\pi.$$

(14), (15), and (16) imply that

$$(17) \quad \|C_\varphi\|_e \geq \limsup_{a \rightarrow \Gamma_j} \frac{N_\varphi(a)}{g_\Omega(a; t_0)}$$

which together with (7) proves the theorem.

To see that (16) holds, we note first that

$$\lim_{b \rightarrow e^{it}} \frac{g_U(b; 0)}{\frac{1}{2}(1 - |b|^2)} = \lim_{b \rightarrow e^{it}} \frac{g_U(b; 0)}{1 - |b|} = \frac{\partial g_U}{\partial n}(e^{it}) = V(e^{it})$$

where  $V$  is the function such that  $V dt = 2\pi d\omega_0^U$  on the unit circle  $\mathbb{T}$ . Now let  $u$  be any continuous function on  $\mathbb{T}$  and let  $\tilde{u}$  denote its harmonic extension to  $\Delta$  via the Poisson kernel. Then

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathbb{T}} u(t) \left( 1 - \int_S \frac{1 - |z|^2}{|1 - \overline{e^{it}}z|^2} d\omega_0^U \right) dt \\ &= \tilde{u}(0) - \int_S \left( \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{e^{it}}z|^2} u(t) dt \right) d\omega_0^U(z) \\ &= \tilde{u}(0) - \int_S \tilde{u}(z) d\omega_0^U(z) \\ &= \tilde{u}(0) - \int_{\partial U} \tilde{u}(z) d\omega_0^U(z) + \int_{\mathbb{T}} u d\omega_0^U(z) \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} u V dt. \end{aligned}$$

This shows that

$$\begin{aligned} V(e^{it}) &= 1 - \int_S \frac{1 - |z|^2}{|1 - \overline{e^{it}}z|^2} d\omega_0^U \\ &= \lim_{b \rightarrow e^{it}} (1 - I(b)), \end{aligned}$$

which gives us (16), so we are done.

#### REFERENCES

- [1] S. D. Fisher, *Function Theory on Planar Domains*, (1983) John Wiley & Sons, New York.
- [2] S. D. Fisher, *Eigen-values and eigen-vectors of compact composition operators on  $H^p(\Omega)$* , Indiana J. Math, **32** (1983), 843-847.
- [3] I. Kra, *Automorphic Forms and Kleinian Groups*, (1972) W.A. Benjamin, Inc., Reading, Mass..
- [4] J. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Society (2), **23** (1925), 481-519.
- [5] R. Nevanlinna, *Analytic Functions*, (1970) Springer Verlag, New York.

- [6] W. Rudin, *Analytic functions of the class  $H_p$* , Trans. Amer. Math. Soc., **78** (1955), 46-66.
- [7] J. H. Shapiro, *The essential norm of a composition operator*, Annals of Mathematics, **125** (1987), 375-404.
- [8] M. Voichick and L. Zalcman, *Inner and outer functions on Riemann surfaces*, Proc. Amer. Math. Soc., **16** (1965), 1200-1204.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY