KERNELS OF HANKEL OPERATORS AND HYPONORMALITY OF TOEPLITZ OPERATORS.

CAIXING GU AND JONATHAN E. SHAPIRO

ABSTRACT. We give a formula for $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}}$ and describe when $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}}$. We explore the hyponormality of Toeplitz operators whose symbols are of circulant type and some more general types. In addition, we discuss formulas for and estimates of the rank of the self-commutator of a hyponormal Toeplitz operator.

1. INTRODUCTION

Brown and Halmos [2] started the study of the algebraic properties of Toeplitz operators. They showed that the Toeplitz operator T_φ with symbol φ is normal if and only if $\varphi = \alpha u + \beta$ for some real-valued function u and complex numbers α and β. Cowen [3] recently showed that T_φ is hyponormal if and only if there exists $k \in H^\infty$, $\|k\|_\infty \leq 1$, such that $\varphi_- - k\overline{\varphi_+} \in H^2$, where $\varphi_+ = P(\varphi)$ and $\varphi_- = (I-P)(\varphi)$. See also Nakazi and K. Takahashi [17] and the first named author [11] for generalizations and refinements of this characterization.

Recent efforts have been made to give more explicit conditions for the hyponormality of T_φ. That is, how we can actually check if there is such a $k \in H^\infty$, $\|k\|_\infty \leq 1$ such that $\varphi_- - k\overline{\varphi_+} \in H^2$. Particular attention has been paid to Toeplitz operators with polynomial symbols. In Particular, using the results of Nakazi and Takahashi [17], and Zhu [20], Lee and his collaborators [5], [6], [13] and [15] gave explicit conditions for the hyponormality of Toeplitz operators with polynomial symbols whose coefficients satisfy certain symmetric or circulant relations. The main purpose of this paper is to put their results in a much more general and abstract setting and to give explicit criteria for the hyponormality of a broader class of Toeplitz operators. Our approach emphasizes the use of Hankel operators and the proofs are done in a more abstract and often simpler way.

To be more specific, we now outline the plan of the paper. Some questions about the hyponormality of Toeplitz operators are seen to be related to properties of the kernels of Hankel operators and products of Hankel operators. In Section 2, we explore the kernels of products of Hankel operators, leading to the explicit formula for $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}}$. We use this formula to understand better when $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}}$ and to provide a simple proof of the main result of Nakazi in [16].

In Section 3, we use the result mentioned above to tell us about the hyponormality of a Toeplitz operator with circulant-type symbol. This allows us very easily to derive the explicit conditions for hyponormality of a Toeplitz operator when the symbol is a circulant polynomial.

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The results of Section 3 concerning hyponormality of Toeplitz operators with circulant-type symbols are generalized to a broader class of symbols in Section 4. Formulas for and estimates of the rank of the self-commutator of a hyponormal Toeplitz operator are then developed in Section 5.

For Toeplitz operators with symbols of bounded type, we reduce the determination of hyponormality to the computation of the norm of certain Hankel operators. This takes place in Section 6.

Finally, in Section 7, we use the computation of Hankel operator norms to give more explicit examples of hyponormal Toeplitz operators with symbols satisfying some symmetric conditions. It is a remarkable fact that the Hankel operators we are interested in here have been studied extensively in the recent literature because of their important application in robust control theory; see the book [9] by Foias, Özbay and Tannenbaum for details and more references. In particular, we indicate an efficient algorithm from [12] by the first named author, Toker and Özbay for the computation of Hankel operator norms. We illustrate this algorithm by a self-contained discussion in a simple case. We then show how some conditions on the coefficients of the polynomial symbols of hyponormal Toeplitz operators in [5] and [15] follow from this simple case, and we conclude the paper with an explicit example of hyponormal Toeplitz operators with irrational symbols.

1.1. **Notation and Preliminary Results.** For φ in L^∞, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined by $T_\varphi h = P(\varphi h)$, where P is the projection operator from L^2 to H^2. Let $J : \left(H^2\right)^\perp \to H^2$ be given by $Je^{-in\theta} = e^{i(n-1)\theta}$ for $n \geq 1$. The Hankel operator $H_f : H^2 \to H^2$ will be defined by $H_f h = J(I-P)(fh)$. Let $S = T_z$ be the shift. It is then true that $H_f S = S^* H_f$. Also, for $f \in L^\infty$, we define $\widetilde{f} = \overline{f(\overline{z})}$, and note that $H_f^* = H_{\widetilde{f}}$. We can then verify, using the definition of the Hankel operator, that, for $g \in H^\infty$,

$$H_f T_g = H_{fg} = T_{\widetilde{g}}^* H_f. \tag{1.1}$$

Using (1.1) and the connection between Hankel and Toeplitz operators

$$T_{\varphi\psi} - T_\varphi T_\psi = H_{\overline{\varphi}}^* H_\psi, \ \text{ where } \varphi, \psi \in L^\infty,$$

we can show

Lemma 1. *For an inner function θ and $\varphi \in L^\infty$,*

$$H_\varphi^* H_\varphi - H_{\theta\varphi}^* H_{\theta\varphi} = H_\varphi^* \left(1 - T_{\widetilde{\theta}} T_{\widetilde{\theta}}^*\right) H_\varphi = H_\varphi^* H_{\overline{\widetilde{\theta}}}^* H_{\overline{\widetilde{\theta}}} H_\varphi = H_\varphi^* H_{\overline{\theta}} H_{\overline{\theta}}^* H_\varphi.$$

For an inner function θ, $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$. We will make use of the characterization of $\mathcal{H}(\theta)$:

$$\mathcal{H}(\theta) = \left\{ f \in H^2 | \overline{\theta} f \in \overline{H_0^2} \right\}.$$

It is then easy to show

Lemma 2. *For an inner function θ, $\overline{z}\theta\overline{\mathcal{H}(\theta)} = \mathcal{H}(\theta)$.*

Proof. If $f \in \mathcal{H}(\theta)$, then $\overline{z}\theta\overline{f} \in \mathcal{H}(\theta)$ since $\overline{\theta}\overline{z}\theta\overline{f} = \overline{z}\overline{f} \in \overline{H_0^2}$, thus $\overline{z}\theta\overline{\mathcal{H}(\theta)} \subset \mathcal{H}(\theta)$. From this inclusion, we can multiply both sides by $z\overline{\theta}$ and take conjugates to get $\mathcal{H}(\theta) \subset \overline{z}\theta\overline{\mathcal{H}(\theta)}$. ∎

We will also frequently make use of the facts that $\ker H_{\overline{\theta}} = \theta H^2$, and thus range $\left(H_{\overline{\theta}}^*\right) = \mathcal{H}(\theta)$.

2. Hankel Operator Kernels

It has been shown by the first named author that

Theorem 1 (Gu, 1999)**.** *For any two L^∞ functions f and g,*

$$\text{either} \quad \ker H_f^* H_g = \ker H_g \ \text{ or } \ \ker H_g^* H_f = \ker H_f .$$

See [10] for a proof of this theorem and more general results for products of finite many Hankel operators. As a corollary of this theorem, we get a result of Axler, Chang, and Sarason in [1], which can be stated as

Theorem 2 (Axler, Chang, and Sarason, 1978)**.** *$H_f^* H_g$ has finite rank exactly when at least one of $(I-P)f$ and $(I-P)g$ is a rational function. In this case, the rank of $H_f^* H_g$ is the minimum of the degrees of $(I-P)f$ and $(I-P)g$.*

Proof. $H_f^* H_g$ has finite rank iff $H_g^* H_f$ does, too, and by Theorem 1, this rank, which is the codimension of the kernel, is either the rank of H_g or H_f, at least one of which is finite. Since the rank of $H_g^* H_f$ is at most the rank of H_f and the rank of $H_f^* H_g$ (which must be the same) is at most the rank of H_g, this rank must be the minimum of the ranks of H_g and H_f, which is the minimum of the degrees of $(I-P)f$ and $(I-P)g$. ∎

We now wish to understand better the when $\ker H_f^* H_g = \ker H_g$ and when $\ker H_g^* H_f = \ker H_f$, particularly in the case where f and g are conjugates of inner functions.

Theorem 3. *For any two inner functions θ_1 and θ_2,*

$$\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}} = \theta_2 H^2 \oplus \theta_2 \overline{z(\theta_1 H^2 \cap \mathcal{H}(\theta_2))}.$$

Proof. $\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}} \supset \ker H_{\overline{\theta_2}} = \theta_2 H^2$, so for any p in $\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}}$ which is not in $\theta_2 H^2$, $p \in \ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}}$ is equivalent to $0 \neq H_{\overline{\theta_2}} p \in \ker H_{\overline{\theta_1}}^* = J\left(\overline{z}\overline{\theta_1 H^2}\right)$, or

$$\overline{\theta_2} p = \overline{z}\overline{\theta_1 h} + k$$

for some H^2 functions h and k, with $h \neq 0$. We rewrite this as

$$p = \overline{z}\overline{\theta_1 h}\theta_2 + \theta_2 k.$$

Now consider the orthogonal projection of p onto the space $\mathcal{H}(\theta_2)$. This projection is given by $\overline{z}\overline{\theta_1 h}\theta_2$, since $\overline{z}\overline{\theta_1 h}\theta_2$ can be seen to be in $\mathcal{H}(\theta_2)$, while $\theta_2 k$ is orthogonal to $\mathcal{H}(\theta_2)$. This tells us that $\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}}$ consists of the orthogonal sum of $\theta_2 H^2$ and elements of the form $\overline{z}\overline{\theta_1 h}\theta_2$ which are in H^2, i.e.,

$$\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}} = \theta_2 H^2 \oplus \theta_2 \overline{z}\overline{\theta_1 H^2} \cap H^2.$$

Since $\theta_2 \overline{z}\overline{\theta_1 H^2} \subset \mathcal{H}(\theta_2)$, we can write

$$\begin{aligned}
\ker H_{\overline{\theta_1}}^* H_{\overline{\theta_2}} &= \theta_2 H^2 \oplus \theta_2 \overline{z}\overline{\theta_1 H^2} \cap \mathcal{H}(\theta_2) \\
&= \theta_2 H^2 \oplus \theta_2 \overline{z}\overline{\theta_1 H^2} \cap \overline{z}\theta_2 \overline{\mathcal{H}(\theta_2)} \qquad \text{(by Lemma 2)} \\
&= \theta_2 H^2 \oplus \theta_2 \overline{z}\left(\overline{\theta_1 H^2} \cap \overline{\mathcal{H}(\theta_2)}\right) \\
&= \theta_2 H^2 \oplus \theta_2 \overline{z(\theta_1 H^2 \cap \mathcal{H}(\theta_2))}.
\end{aligned}$$

∎

From this, we get:

Corollary 1. *For any two inner functions θ_1 and θ_2,*

$$\text{closure}\left(\text{range}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right)\right) = \theta_2 \overline{z} \overline{(\mathcal{H}(\theta_2) \ominus (\theta_1 H^2 \cap \mathcal{H}(\theta_2)))}.$$

Proof.

$$\begin{aligned}
\text{closure}\left(\text{range}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right)\right) &= \left(\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}}\right)^{\perp} \\
&= \left(\theta_2 H^2 \oplus \theta_2 \overline{z} \overline{(\theta_1 H^2 \cap \mathcal{H}(\theta_2))}\right)^{\perp} \\
&= \mathcal{H}(\theta_2) \ominus \theta_2 \overline{z} \overline{(\theta_1 H^2 \cap \mathcal{H}(\theta_2))} \\
&= \theta_2 \overline{z} \overline{\mathcal{H}(\theta_2)} \ominus \theta_2 \overline{z} \overline{(\theta_1 H^2 \cap \mathcal{H}(\theta_2))} \\
&= \theta_2 \overline{z} \overline{(\mathcal{H}(\theta_2) \ominus (\theta_1 H^2 \cap \mathcal{H}(\theta_2)))}.
\end{aligned}$$

The question of when $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}}$ can now be answered as follows. ∎

Corollary 2. *For inner functions θ_1 and θ_2, $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}}$ if and only if $\theta_1 H^2 \cap \mathcal{H}(\theta_2) = \{0\}$.*

Proof. By Theorem 3, $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \theta_2 H^2 \oplus \theta_2 \overline{z} \overline{(\theta_1 H^2 \cap \mathcal{H}(\theta_2))}$, so $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}} = \theta_2 H^2$ when $\theta_1 H^2 \cap \mathcal{H}(\theta_2) = \{0\}$. ∎

We can use the results above to provide a different proof of the main theorem of Nakazi from [16],

Theorem 4 (Nakazi, 1987)**.** *Let n be a nonnegative integer. Let θ_1 and θ_2 be inner functions. Suppose that $\mathcal{H}(\theta_2) \cap \theta_1 H^2 \neq \{0\}$. Then the dimension of $\mathcal{H}(\theta_2) / \left(\mathcal{H}(\theta_2) \cap \theta_1 H^2\right)$ is n if and only if θ_1 is a Blaschke product of degree n.*

Proof. The hypothesis that $\mathcal{H}(\theta_2) \cap \theta_1 H^2 \neq \{0\}$, by the corollary above, implies that $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} \neq \ker H_{\overline{\theta_2}}$. By Theorem 1, $\ker H^*_{\overline{\theta_2}} H_{\overline{\theta_1}} = \ker H_{\overline{\theta_1}}$, that is $\text{closure}\left(\text{range}\left(H^*_{\overline{\theta_1}} H_{\overline{\theta_2}}\right)\right) = \text{Range}\left(H^*_{\overline{\theta_1}}\right) = \mathcal{H}(\theta_1)$.

By Corollary 1, $\dim \mathcal{H}(\theta_2) / \left(\mathcal{H}(\theta_2) \cap \theta_1 H^2\right) = n$ exactly when

$$\dim\left[\text{closure}\left(\text{range}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right)\right)\right] = \text{rank}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right) = n,$$

or, equivalently,

$$\text{rank}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right) = \text{rank}\left(H^*_{\overline{\theta_1}} H_{\overline{\theta_2}}\right) = \dim \mathcal{H}(\theta_1) = n.$$

This happens exactly when θ_1 is a Blaschke product of degree n. ∎

To understand better when $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} = \ker H_{\overline{\theta_2}}$ and when $\ker H^*_{\overline{\theta_2}} H_{\overline{\theta_1}} = \ker H_{\overline{\theta_1}}$, we define a partial order on the set of all inner functions as follows:

Definition 1. $\theta_1 \prec \theta_2$ *if* $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} \neq \ker H_{\overline{\theta_2}}$, *i.e.,* $\theta_1 H^2 \cap \mathcal{H}(\theta_2) \neq \{0\}$.

This does, in fact, give us a partial order since if $\theta_1 \prec \theta_2$ and $\theta_2 \prec \theta_3$, then there are functions h_1 and h_2 in H^2 such that $\theta_1 h_1 \in \mathcal{H}(\theta_2)$ and $\theta_2 h_2 \in \mathcal{H}(\theta_3)$. But then $\overline{\theta_2}\theta_1 h_1 \in \overline{H_0^2}$, and $\overline{\theta_3}\theta_2 h_2 \in \overline{H_0^2}$, so $\left(\overline{\theta_3}\theta_2 h_2\right)\left(\overline{\theta_2}\theta_1 h_1\right) = \overline{\theta_3}\theta_1 h_1 h_2 \in \overline{H_0^2}$, i.e., $\theta_1 h_1 h_2 \in \mathcal{H}(\theta_3)$, or $\theta_1 \prec \theta_3$.

Proposition 1. *The order $\prec$ is a strict partial order.*

Proof. By Theorem 1, it is impossible to have both $\ker H^*_{\overline{\theta_1}} H_{\overline{\theta_2}} \neq \ker H_{\overline{\theta_2}}$ and $\ker H^*_{\overline{\theta_2}} H_{\overline{\theta_1}} \neq \ker H_{\overline{\theta_1}}$, i.e., $\theta_1 \prec \theta_2$ and $\theta_2 \prec \theta_1$. ∎

Exactly when $\theta_1 \prec \theta_2$ is still not fully understood, but there are several things we can say.

1. If $z\theta_1 | \theta_2$, then $\theta_1 \prec \theta_2$.

   Proof. If $z\theta_1 | \theta_2$, then $\overline{z\theta_1}\theta_2 \in H^2$, so $\overline{\theta_2}\theta_1 \in \overline{H_0^2}$, and thus $\theta_1 \prec \theta_2$. ∎

2. If $\deg \theta_1 > \deg \theta_2$ for finite Blaschke products θ_1 and θ_2, or for an infinite Blaschke product θ_1 (for which we say that $\deg \theta_1 = \infty$) and a finite Blaschke product θ_2, then $\theta_1 \not\prec \theta_2$.

   Proof. We have an explicit representation in these cases for $\mathcal{H}(\theta_2)$ (see [7, page 278]) which tells us that if θ_2 is a finite Blaschke product, then $\mathcal{H}(\theta_2)$ contains members whose inner factors can be any finite Blaschke product of degree smaller than $\deg \theta_2$, and if θ_2 is an infinite Blaschke product, then $\mathcal{H}(\theta_2)$ contains members whose inner factor is any desired finite Blaschke product. ∎

3. If $\deg \theta_1 < \deg \theta_2$ (as in (2)), then $\theta_1 \prec \theta_2$.

   Proof. By Proposition 1 and the previous fact. ∎

4. If $\deg \theta_1 = \deg \theta_2$ is finite, then $\theta_1 \not\prec \theta_2$.

   Proof. Again, by the explicit representation of $\mathcal{H}(\theta_2)$, we see that the degree of the Blaschke factor of any element $\mathcal{H}(\theta_2)$ is smaller than $\deg \theta_2 = \deg \theta_1$. ∎

5. If $\theta_1 = \theta_1'\theta$ and $\theta_2 = \theta_2'\theta$ for some inner function θ, then $\theta_1 \prec \theta_2$ if and only if $\theta_1' \prec \theta_2'$.

   Proof. $\theta_1 \prec \theta_2$ if and only if $\overline{\theta_2}\theta_1 h \notin \overline{H_0^2}$ for any $h \in H^2$, if and only if $\overline{\theta_2'}\overline{\theta}\theta\theta_1' h \notin \overline{H_0^2}$ for any $h \in H^2$, if and only if $\theta_1' \prec \theta_2'$. ∎

Note that since the closure of the range of an operator is the space perpendicular to the kernel of its adjoint, we get

Corollary 3. *$\theta_1 \not\prec \theta_2$ if and only if* closure $\left(\text{range}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_1}}\right)\right) = \text{range}\left(H^*_{\overline{\theta_2}}\right) = \mathcal{H}(\theta_2)$.

3. Hyponormality of $T_{f+\overline{\theta}f}$

In this section we discuss the hyponormality of Toeplitz operators with symbols satisfying a certain circulant symmetry. We first recall Cowen's characterization of hyponormal Toeplitz operators [3]; see also [11] and [17] for generalizations and refinements of this characterization. For $\varphi \in L^\infty$, let $\varphi = \varphi_+ + \overline{\varphi_-}$ be the decomposition of φ into its analytic and co-analytic parts.

Cowen's Theorem (1988). *T_φ is hyponormal if and only if there exist $k \in H^\infty$, $\|k\|_\infty \leq 1$ such that*

$$\overline{\varphi_-} - k\overline{\varphi_+} \in H^2.$$

A slight variant of the above characterization in [17] is that

$$\varphi - k\overline{\varphi} \in H^{\infty},$$

which deals with only bounded functions.

Proposition 2. *Let θ, θ_1, and θ_2 be inner. Assume $\theta \not\leq \theta_1\theta_2$. Then*

$$H^*_{\overline{\theta}\theta_1} H_{\overline{\theta}\theta_1} - H^*_{\overline{\theta}\theta_2} H_{\overline{\theta}\theta_2} \geq 0$$

if and only if $\theta_1|\theta_2$.

Proof. First, by Corollary 3, since $\theta \not\leq \theta_1\theta_2$,

$$\text{closure}\left(\text{range}\left(H^*_{\overline{\theta_1\theta_2}} H_{\overline{\theta}}\right)\right) = \text{range}\left(H^*_{\overline{\theta_1\theta_2}}\right) = \mathcal{H}\left(\theta_1\theta_2\right). \tag{3.1}$$

We wish to know when

$$\begin{aligned} H^*_{\overline{\theta}\theta_1} H_{\overline{\theta}\theta_1} - H^*_{\overline{\theta}\theta_2} H_{\overline{\theta}\theta_2} &= H^*_{\overline{\theta}}\left(T_{\theta_1}T^*_{\theta_1} - T_{\theta_2}T^*_{\theta_2}\right) H_{\overline{\theta}} \text{ (by (1.1))} \\ &= H^*_{\overline{\theta}}\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_2}} - H^*_{\overline{\theta_1}} H_{\overline{\theta_1}}\right) H_{\overline{\theta}} \geq 0 \end{aligned} \tag{3.2}$$

holds.

$H^*_{\overline{\theta}} H_{\overline{\theta}}$ is the projection onto $\mathcal{H}(\theta)$, so range $\left(H^*_{\overline{\theta_2}} H_{\overline{\theta_2}} - H^*_{\overline{\theta_1}} H_{\overline{\theta_1}}\right) \subset \mathcal{H}(\theta_1) + \mathcal{H}(\theta_2) \subset \mathcal{H}(\theta_1\theta_2)$. (3.2) holds iff

$$\left\langle \left(H^*_{\overline{\theta_2}} H_{\overline{\theta_2}} - H^*_{\overline{\theta_1}} H_{\overline{\theta_1}}\right) P_{\mathcal{H}(\theta_1\theta_2)} H_{\overline{\theta}} h,\ P_{\mathcal{H}(\theta_1\theta_2)} H_{\overline{\theta}} h \right\rangle \geq 0 \text{ for all } h \in H^2.$$

By assumption (3.1), closure $\left(P_{\mathcal{H}(\theta_1\theta_2)}\left(H_{\overline{\theta}} H^2\right)\right) = \mathcal{H}(\theta_1\theta_2)$, so

$$H^*_{\overline{\theta_2}} H_{\overline{\theta_2}} - H^*_{\overline{\theta_1}} H_{\overline{\theta_1}} \geq 0.$$

This is equivalent to $\ker H_{\overline{\theta_1}} = \theta_1 H^2 \supset \ker H_{\overline{\theta_2}} = \theta_2 H^2$, or $\theta_1|\theta_2$. ∎

Let $\varphi(z) = f(z) + \overline{\theta} f(z)$ for some inner function θ and $f \in \mathcal{H}(\theta)$. Let $\varphi = \varphi_+ + \overline{\varphi_-}$ be the decomposition of φ into its analytic and co-analytic parts. Define $f^\sharp(z) = \theta \overline{f} \in H^2$.

$$\overline{\varphi_+} = \overline{f} = \overline{\theta}\left(\theta\overline{f}\right) = \overline{\theta} f^\sharp, \qquad \overline{\varphi_-} = \overline{\theta} f \tag{3.3}$$

Note that since $f^\sharp$ and f have the same outer part, say f_0, we can write $f^\sharp(z) = \theta_1' f_0(z)$ and $f(z) = \theta_2' f_0(z)$, from which we can write

$$\frac{f^\sharp(z)}{f(z)} = \theta_1 \overline{\theta_2}$$

or

$$\theta_1 f = \theta_2 f^\sharp, \tag{3.4}$$

where θ_1 and θ_2 are relatively prime inner functions, with $\theta_1|\theta_1'$ and $\theta_2|\theta_2'$. Without loss of generality, we can assume that $\theta_1\theta_2$ is relatively prime to θ.

Theorem 5. *If $\theta \not\leq \theta_1\theta_2$, then $T_{f+\overline{\theta}f}$ is hyponormal iff $\theta_1|\theta_2$.*

Proof.

$$\begin{aligned} \left[T^*_{\varphi}, T_{\varphi}\right] &= T^*_{\varphi} T_{\varphi} - T_{\varphi} T^*_{\varphi} = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_{\varphi} H_{\varphi} \\ &= H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}} = H^*_{\overline{\theta} f^\sharp} H_{\overline{\theta} f^\sharp} - H^*_{\overline{\theta} f} H_{\overline{\theta} f} \geq 0 \end{aligned}$$

(by Cowen's Theorem) $\Longleftrightarrow$ there is some $k \in H^\infty$, $\|k\|_\infty \leq 1$ and $h \in H^2$ with

$$\overline{\theta} f - k\overline{\theta} f^\sharp = h$$

which is equivalent to

$$f - kf^\sharp = \theta h$$

or

$$\begin{aligned} f - k\theta_1\overline{\theta_2} f &= \theta h, \\ f\,(\theta_2 - k\theta_1) &= \theta_2\theta h. \end{aligned}$$

Since θ and f are relatively prime, we must have $\theta_2 - k\theta_1 = \theta h$, or $\overline{\theta}\theta_2 - k\overline{\theta}\theta_1 = h$. By Cowen's Theorem again, this happens if and only if

$$H^*_{\overline{\theta}\theta_1} H_{\overline{\theta}\theta_1} - H^*_{\overline{\theta}\theta_2} H_{\overline{\theta}\theta_2} \geq 0.$$

By Proposition 2, this happens if and only if $\theta_1 | \theta_2$. ∎

3.1. **Example.** We consider the case where φ is a circulant polynomial, as in [6]. We will see that the theorems above lead to a simpler and more natural proof of the characterization of when a Toeplitz operator with circulant polynomial symbol is hyponormal. We have the definition of a circulant polynomial,

Definition 2. *A trigonometric polynomial* $\varphi(z) = \sum_{n=-m}^{N} c_n z^n$ *of analytic degree* N *and co-analytic degree* m *is said to be a* circulant polynomial with argument ω *if (i)* $m \leq N$ *and (ii) there exists* $\omega \in [0, 2\pi)$ *such that* $c_{-k} = e^{i\omega} c_{N-k+1}$ *for every* $1 \leq k \leq m$*., and (iii)* $c_0 = c_1 = \cdots = c_{N-m} = 0$ *when* $m < N$.

For such a φ, it is easy to see that we can write

$$\varphi(z) = c_N z^N + \cdots + c_{N-m+1} z^{N-m+1} + e^{i\omega} c_N z^{-1} + + \cdots e^{i\omega} c_{N-m+1} z^{-m}$$

or

$$\varphi(z) = z^{-m}(z^{N+1} + e^{i\omega}) g(z) = f(z) + e^{i\omega} z^{-N-1} f(z) \tag{3.5}$$

where

$$g(z) = c_N z^{m-1} + c_{N-1} z^{m-2} + \cdots + c_{N-m+1} \quad \text{and} \quad f(z) = z^{N-m+1} g(z).$$

We are now in the situation where $\varphi = f + \overline{\theta} f$, and $f \in \mathcal{H}(\theta)$, since $\theta = e^{-i\omega} z^{N+1}$ has degree $N+1$, and f is a polynomial of degree N.

T_φ is thus hyponormal if and only if $T_{f+\overline{\theta} f}$ is hyponormal, which happens if and only if $\theta_1 | \theta_2$, where θ_1 and θ_2 are the relatively prime parts of the inner factors of $f^\sharp = \theta \overline{f}$, and f, respectively. This is a consequence of Theorem 5, since it is easy to see that deg $(\theta_1\theta_2) \leq$ deg $(\theta) = N + 1$, i.e., $\theta \not\prec \theta_1\theta_2$.

θ_1 is the inner factor of $z^{m-1}\overline{g(z)} = \overline{c_N} + \overline{c_{N-1}} z + \cdots + \overline{c_{N-m+1}} z^{m-1}$, so $\theta_1(z)$ is zero in the unit disk $\mathbb{D}$ exactly at those z where $g(1/\overline{z}) = 0$. θ_2 is the inner factor of g, so it is zero in $\mathbb{D}$ where g is zero. $\theta_1 | \theta_2$ exactly when, for each root ζ of g with $|\zeta| < 1$, $1/\overline{\zeta}$ is a root of g of multiplicity at least as great as the multiplicity of ζ. This is exactly the criterion for hyponormality of the Toeplitz operator T_φ, with φ a circulant polynomial, which was proved in Farenick and Lee [6].

3.2. **Example.** The above result of [6] has been generalized in [13] as follows

Theorem 6 (Hwang, Kim, and Lee, 1999). *Suppose* $\varphi(z) = \sum_{n=-m}^{N} c_n z^n$ *with* $m \leq N$ *and write*

$$\Im := \{\zeta, 1/\overline{\zeta} : \text{ the complex numbers } \zeta \text{ and } 1/\overline{\zeta} \text{ are zeros of } z^m\varphi(z)\}.$$

If $\Im$ *contains at least* $(N+1)$ *elements then the following statement are equivalent.*

(i) T_φ *is hyponormal.*

(ii) For every zero ζ *of* $z^m\varphi(z)$ *such that* $|\zeta| > 1$, *the number* $1/\overline{\zeta}$ *is a zero of* $z^m\varphi(z)$ *in the open unit disk of multiplicity greater than or equal to the multiplicity of* ζ.

We next show how to derive this generalization also from Theorem 5 by using our notation and Cowen's characterization. It follows from the assumption that

$$z^m\varphi(z) = p(z)q(z)$$

where $p(z)$ is the polynomial of degree $N+1$ whose zeros consist of $N+1$ elements from $\Im$ such that if ζ is a zero then $1/\overline{\zeta}$ is also a zero of $p(z)$. Thus $q(z)$ is a polynomial of degree $m-1$. It is easy to verify that $p(z)$ satisfies the relation

$$p^\sharp(z) := z^{N+1}\overline{p(z)} = p(z).$$

We note that the previous example is a special case of this example since by (3.5), $z^m\varphi(z) = (z^{N+1} + e^{i\omega})g(z)$. By a variant of Cowen's Theorem, as noted in [17], T_φ is hyponormal if and only if there is some $k \in H^\infty$, $\|k\|_\infty \leq 1$ and $h \in H^2$ with

$$\varphi - k\overline{\varphi} = h.$$

Multiplying both sides by z^m, we have

$$p(z)q(z) - kz^{N+1}\overline{p(z)}z^{m-N-1}\overline{q(z)} = p(z)\left[q(z) - kz^{m-N-1}\overline{q(z)}\right] = z^m h.$$

Equivalently

$$\overline{z}^m q(z) - k\overline{z^{N+1}q(z)} = h_1$$

for some $h_1 \in H^2$. By Cowen's Theorem again, this happens if and only if T_ψ is hyponormal where

$$\psi = z^{N+1}q(z) + \overline{z}^m q(z) = f(z) + \overline{z}^{N+1+m} f(z) = f(z) + \overline{\theta} f(z).$$

with $\theta = z^{N+1+m}$. Note that ψ is of circulant type exactly as defined above. The result follows now from Theorem 5 as explained above.

4. Hyponormality of T_φ with $\|\varphi_+\|_2 = \|\varphi_-\|_2$

Let $\varphi \in L^2$ of the unit circle, and write $\varphi = \varphi_+ + \overline{\varphi_-}$, where φ_+ is the analytic part of φ, i.e., the projection of φ onto H^2, and $\overline{\varphi_-}$ is the anti-analytic part of φ, i.e., $\varphi_- \in H_0^2$. In this section we characterize hyponormal Toeplitz operators T_φ with symbols φ satisfying $\|\varphi_+\|_2 = \|\varphi_-\|_2$. As a consequence we get a strengthened version of Theorem 5.

Theorem 7. *For* $\varphi = \varphi_+ + \overline{\varphi_-}$ *as above, if* $\|\varphi_+\|_2 = \|\varphi_-\|_2$, *then the Toeplitz operator* T_φ *is hyponormal if and only if* $\varphi_- | \varphi_+$ *(in* H^2*).*

Proof. By Cowen's Theorem, T_φ is hyponormal if and only if there is some $k \in H^\infty$ with $\|k\|_\infty \le 1$, and $h \in H^2$ with

$$\overline{\varphi_-} - k\overline{\varphi_+} = h. \tag{4.1}$$

When this happens, we have

$$\begin{aligned} \|\varphi_-\|_2 &= \|\overline{\varphi_-}\|_2 = \left\|P_{\overline{H_0^2}}\overline{\varphi_-}\right\|_2 = \left\|P_{\overline{H_0^2}}k\overline{\varphi+}\right\|_2 \\ &\le \|k\overline{\varphi+}\|_2 \le \|k\|_\infty \|\overline{\varphi+}\|_2 \le \left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2. \end{aligned}$$

We can thus conclude that we have equality everywhere in the above, in particular, that k must be an inner function, and $\left\|P_{\overline{H_0^2}}k\overline{\varphi+}\right\|_2 = \|k\overline{\varphi+}\|_2$, i.e., that $k\overline{\varphi+} \in \overline{H_0^2}$, so $\overline{\varphi_-} - k\overline{\varphi_+} \in \overline{H_0^2}$, and thus $h = 0$. We then have

$$\overline{\varphi_-} = k\overline{\varphi_+}.$$

Since k is an inner function, we can write

$$k\varphi_- = \varphi_+,$$

or $\varphi_-|\varphi_+$.

In the opposite direction, it is clear that if $\varphi_-|\varphi_+$, then $\theta\varphi_- = \varphi_+$, and (4.1) holds, with $h = 0$, and $k = \theta$. ∎

Corollary 4. *For $\varphi = \varphi_+ + \overline{\varphi_-}$ as above, if $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$, then the Toeplitz operator T_φ is hyponormal if and only if $\varphi = k\overline{\varphi}$ for some inner function k.*

Proof. T_φ is hyponormal iff $k\varphi_- = \varphi_+$ for some inner function k, by the proof above. For $\varphi = \varphi_+ + \overline{\varphi_-} = k\varphi_- + \overline{\varphi_-}$, $\overline{\varphi} = \overline{k\varphi_-} + \varphi_-$, so $k\overline{\varphi} = \theta\left(\overline{k\varphi_-} + \varphi_-\right) = k\varphi_- + \overline{\varphi_-} = \varphi$. ∎

Corollary 5. *If $\varphi = f + \overline{\theta}f$, with notation as in (3.3) and (3.4), then T_φ is hyponormal if and only if $\theta_1|\theta_2$.*

Proof. $\varphi_+ = f$, and $\varphi_- = f^\sharp$. $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$, so T_φ is hyponormal if and only $\varphi_-|\varphi_+$, i.e., $f^\sharp|f$, or $\theta_1|\theta_2$. ∎

This generalizes Theorem 5, since we no longer need the condition that $\theta \not\leq \theta_1\theta_2$ in order to get the conclusion that T_φ is hyponormal if and only if $\theta_1|\theta_2$.

If $\varphi \in L^2$ satisfies $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$ and T_φ is hyponormal, then, as noted in Corollary 4, we can write, $k\varphi_- = \varphi_+$ for some inner function k. This means that φ_+ and φ_- have the same outer factor, say f, and we can write $\varphi_- = \theta_1 f$ and $\varphi_+ = k\theta_1 f$ for some inner function θ_1. The general form of the symbol φ of a hyponormal Toeplitz operator T_φ for which $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$ is

$$\varphi = \varphi_+ + \overline{\varphi_-} = k\theta_1 f + \overline{\theta_1 f}.$$

One consequence of the above is that if $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$, and T_φ is hyponormal, then, since $\varphi_-|\varphi_+$, we must have $\varphi_+ \in H_0^2$. If $\varphi = \varphi_+ + c + \overline{\varphi_-}$ where $\varphi_+, \varphi_- \in H_0^2$, and c is a constant, then it is easy to see that T_φ is hyponormal if and only if $T_{\varphi - c} = T_{\varphi_+ + \overline{\varphi_-}}$ is hyponormal, so if $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$, then T_φ is hyponormal if and only if $\varphi_-|\varphi_+$.

5. Rank of $\left[T_\varphi^*, T_\varphi\right]$

It is relevant to mention the following characterization of hyponormal Toeplitz operators whose self-commutator $\left[T_\varphi^*, T_\varphi\right]$ is of finite rank [17].

Theorem (Nakazi and Takahashi, 1993). *T_φ is hyponormal and the self-commutator $\left[T_\varphi^*, T_\varphi\right]$ is of finite rank if and only if there exists a finite Blaschke product k such that $\varphi - k\overline{\varphi} \in H^\infty$ and moreover one can chose a k such that the degree of k is equal to the rank of $\left[T_\varphi^*, T_\varphi\right]$.*

It is possible to have different k such that $\varphi - k\overline{\varphi} \in H^\infty$, see [13, page 252] for a simple example. To estimate the rank of $\left[T_\varphi^*, T_\varphi\right]$ using the above result, one has to carefully find or argue for the right or unique finite Blaschke product, as was done in [6], [13] and [15].

Here we offer a slightly simpler approach. Let k be any inner function such that $\varphi - k\overline{\varphi} \in H^\infty$(or $\overline{\varphi_-} - k\overline{\varphi_+} \in H^2$). We can write

$$\begin{aligned}\left[T_\varphi^*, T_\varphi\right] &= H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_{k\overline{\varphi}}^* H_{k\overline{\varphi}} \\ &= H_{\overline{\varphi_+}}^* H_{\overline{k}} H_{\overline{k}}^* H_{\overline{\varphi_+}} \quad \text{by Lemma 1.}\end{aligned}$$

We thus have

$$\begin{aligned}\operatorname{rank}\left[T_\varphi^*, T_\varphi\right] &= \operatorname{rank} H_{\overline{\varphi_+}}^* H_{\overline{k}} H_{\overline{k}}^* H_{\overline{\varphi_+}} = \operatorname{rank} H_{\overline{\varphi_+}}^* H_{\overline{k}} \\ &= \min\left\{\operatorname{rank} H_{\overline{\varphi_+}}, \operatorname{rank} H_{\overline{k}}\right\}\end{aligned}$$

by the theorem of Axler, Chang, and Sarason [1].

5.1. **Example.** If T_φ is hyponormal and $\left\|\varphi_+\right\|_2 = \left\|\varphi_-\right\|_2$, then we must have $\varphi_+ = k\varphi_-$. It is then clear that $\operatorname{rank} H_{\overline{\varphi_+}} \geq \operatorname{rank} H_{\overline{k}}$, so

$$\operatorname{rank}\left[T_\varphi^*, T_\varphi\right] = \min\left\{\operatorname{rank} H_{\overline{\varphi_+}}, \operatorname{rank} H_{\overline{k}}\right\} = \operatorname{rank} H_{\overline{k}}$$

5.2. **Example.** If $\varphi = f + \overline{\theta} f$ and T_φ is hyponormal, we have

$$\begin{aligned}\operatorname{rank}\left[T_\varphi^*, T_\varphi\right] &= \operatorname{rank}\left[T_{f+\overline{\theta}f}^*, T_{f+\overline{\theta}f}\right] \\ &= \min\left\{\operatorname{rank} H_{\overline{f+\overline{\theta}f}}^*, \operatorname{rank} H_{\overline{u}}\right\} \\ &= \min\left\{\operatorname{rank} H_{\overline{f}}, \operatorname{rank} H_{\overline{u}}\right\} = \operatorname{rank} H_{\overline{u}},\end{aligned}$$

where u is the inner function defined by $f + \overline{\theta} f = u\overline{f + \overline{\theta} f}$, or, equivalently, $uf^\sharp = f$, i.e., $u = \frac{f}{f^\sharp}$. The fourth equality above comes from the fact that u must be a factor of f, and thus $\operatorname{rank} H_{\overline{u}} \leq \operatorname{rank} H_{\overline{f}}$.

In the example considered earlier, where φ is a circulant polynomial, $\frac{f}{f^\sharp} = z^{N-m}\frac{g(z)}{z^{m-1}\overline{g(z)}} = u$, so we see that u must be a Blaschke product with degree $N - m + Z_{\mathbb{D}} - Z_{C\setminus\mathbb{D}}$, where $Z_{\mathbb{D}}$ is the number of zeros of g in $\mathbb{D}$ and $Z_{C\setminus\mathbb{D}}$ is the number of zeros of g outside of $\mathbb{D}$ (which is the number of zeros of $z^{m-1}\overline{g(z)}$ in $\mathbb{D}$, all of which cancel with the zeros of g when T_φ is hyponormal).

5.3. **Example.** In the more general case as in Theorem 6, let $k_1 = \frac{\overline{\varphi(z)}}{\varphi(z)}$. Write $k_1 = z^{N-m}\frac{z^m\varphi(z)}{z^N\overline{\varphi(z)}}$. Note that the zeros of $z^N\overline{\varphi(z)}$ in $\mathbb{D}$ are the zeros of $z^m\varphi(z)$ outside of $\mathbb{D}$. By Theorem 6, if T_φ is hyponormal, then for every zero ζ of $z^m\varphi(z)$ such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of $z^m\varphi(z)$ in the open unit disk of multiplicity greater than or equal to the multiplicity of ζ. We see that all the zeros of $z^N\overline{\varphi(z)}$ in $\mathbb{D}$ will be canceled by the zeros of $z^m\varphi(z)$. Thus k_1 must be a Blaschke product with degree $N-m+Z_{\mathbb{D}}-Z_{C\setminus\mathbb{D}}$, where $Z_{\mathbb{D}}$ is the number of zeros of $z^m\varphi(z)$ in $\mathbb{D}$ and $Z_{C\setminus\mathbb{D}}$ is the number of zeros of $z^m\varphi(z)$ outside of $\mathbb{D}$ By the assumption that $\Im$ contains at least $(N+1)$ elements and the degree $z^m\varphi(z)$ is $N+m$, it is clear that $Z_{\mathbb{D}} - Z_{C\setminus\mathbb{D}} \le m$.Therefore

$$\begin{aligned}\operatorname{rank}\left[T_\varphi^*, T_\varphi\right] &= \min\left\{\operatorname{rank} H_{\overline{\varphi_+}}, \operatorname{rank} H_{\overline{k_1}}\right\} \\ &= \min\left\{N, N-m+Z_{\mathbb{D}}-Z_{C\setminus\mathbb{D}}\right\} = N-m+Z_{\mathbb{D}}-Z_{C\setminus\mathbb{D}}.\end{aligned}$$

5.4. **Estimates of Rank.** For more general $\varphi \in L^\infty$, if φ is of bounded type (i. e. quotient of two bounded analytic functions), we can still estimate the rank of the self-commutator of T_φ when T_φ is hyponormal.

$$\begin{aligned}\left[T_\varphi^*, T_\varphi\right] &= H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi \\ &= H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} - H_{\overline{\varphi_-}}^* H_{\overline{\varphi_-}} \ge 0.\end{aligned}$$

Since the closure of range $\left(H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}}\right)$ is $\mathcal{H}(\theta_+)$, where θ_+ is an inner function with $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$, we can give an upper bound for the rank of $\left[T_\varphi^*, T_\varphi\right]$ by the dimension of this range, which is $\deg\theta_+$, if θ_+ is a finite Blaschke product. Because of the positivity of $H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} - H_{\overline{\varphi_-}}^* H_{\overline{\varphi_-}}$, we can conclude that $\theta_- | \theta_+$, where θ_- is an inner function with $\ker H_{\overline{\varphi_-}} = \theta_- H^2$. Let $\theta = \theta_+/\theta_-$. It then follows that

$$\operatorname{range}\left[T_\varphi^*, T_\varphi\right] \supset \mathcal{H}(\theta_+) \ominus \mathcal{H}(\theta_-) = \mathcal{H}(\theta\theta_-) \ominus \mathcal{H}(\theta_-) = \theta_- \mathcal{H}(\theta)$$

since $\mathcal{H}(\theta\theta_-) = \mathcal{H}(\theta_-) \oplus \theta_-\mathcal{H}(\theta)$. Thus we have $\operatorname{rank}\left[T_\varphi^*, T_\varphi\right] \ge \dim(\theta_-\mathcal{H}(\theta)) = \deg\theta$, if $\theta = \theta_+/\theta_-$ is a finite Blaschke product.

As an example, consider the case where φ is any trigonometric polynomial, $\varphi(z) = \sum_{n=-m}^{N} c_n z^n$, with T_φ hyponormal (and thus $N \ge m$). We will have $\deg\theta_+ = N$, and $\deg\theta_- = m$, so the results above tell us that

$$N - m \le \operatorname{rank}\left[T_\varphi^*, T_\varphi\right] \le N,$$

which is discussed and proved in [6] (the upper bound is known from [14]).

6. More Hyponormal Toeplitz Operators

In this section we reduce the determination of the hyponormality of Toeplitz operators with symbols of bounded type to the computation of the norm of Hankel operators via the Nehari distance formula. This allows us to construct a class of hyponormal Toeplitz operators with symbols satisfying certain symmetric conditions, including those of polynomial ones discussed in [5] and [15]. This is achieved by using recent results on the efficient computation of the norm of Hankel operators in the context of robust control theory, which will be given in the next section.

Let $\varphi = \varphi_+ + \overline{\varphi_-}$, as earlier, and assume φ is of bounded type. Write $\ker H_{\overline{\varphi_-}} = \theta H^2$ for some inner function θ, and, as noted above, the hyponormality of T_φ implies that there is some inner function θ_0 such that $\ker H_{\overline{\varphi_+}} = \theta_0\theta H^2$. Then we can write

$$\varphi_+ = \theta_0\theta\overline{a}, \qquad \varphi_- = \theta\overline{b}$$

for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$.

By Cowen's Theorem, T_φ will be hyponormal if and only if there are $k \in H^\infty$ with $\|k\|_\infty \le 1$ and $h \in H^2$ with

$$\overline{\varphi_-} - k\overline{\varphi_+} = h$$

or

$$\begin{aligned} \overline{\theta}b - k\overline{\theta_0\theta}a &= h, \\ \theta_0 b - ka &= \theta_0\theta h. \end{aligned}$$

If there is a solution to this last equation, then we must have $\theta_0|k$, so let $k = \theta_0 k_1$ and write the equation as

$$b - k_1 a = \theta h. \tag{6.1}$$

Let $k_1 \in H^\infty$ be any solution of equation (6.1), i.e., $b - k_1 a = \theta h$ for some $h \in H^2$. We do not place any restriction on $\|k_1\|_\infty$. Then all solutions g to $b - ga = \theta h$ are of the form $g = k_1 + \theta f$ for some $f \in H^\infty$. We can thus conclude that T_φ is hyponormal iff there is some solution g of H^∞ norm at most 1, or $\inf_{f\in H^\infty} \|k_1 + \theta f\|_\infty \le 1$. By Nehari's Theorem [18] or the result in [19]

$$\inf_{f\in H^\infty} \|k_1 + \theta f\|_\infty = \inf_{f\in H^\infty} \left\|\overline{\theta}k_1 + f\right\|_\infty = \left\|H_{\overline{\theta}k_1}\right\|.$$

Thus we have

Theorem 8. *If $\varphi = \theta_0\theta\overline{a} + \overline{\theta}b$, for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$, then T_φ is hyponormal iff for any particular solution $k_1 \in H^\infty$ of equation (6.1), $\left\|H_{\overline{\theta}k_1}\right\| \le 1$.*

Let us now decompose a as

$$a = P_{\mathcal{H}(\theta)}a + \theta a_1 = a_0 + \theta a_1.$$

The component a_1 does not have any effect on $\left\|H_{\overline{\theta}k_1}\right\|$ for solutions k_1 of (6.1), so we get

Corollary 6. *If $\varphi = \theta_0\theta\overline{a} + \overline{\theta}b$, for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$, with $a = P_{\mathcal{H}(\theta)}a + \theta a_1 = a_0 + \theta a_1$, then the hyponormality of T_φ is independent of the component a_1.*

Corollary 7. *If $\varphi = \theta_0\theta\overline{a} + \overline{\theta}b$, for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$, then T_φ is hyponormal iff T_ψ is hyponormal, where $\psi = T_{\overline{\theta_0}}\theta_0\theta\overline{a} + \overline{\theta}b = P(\theta\overline{a}) + \overline{\theta}b = \theta\overline{a_0} + \overline{\theta}b$.*

Proof. Note that the hyponormality of both T_φ and T_ψ is equivalent to $\left\|H_{\overline{\theta}k_1}\right\| \le 1$ where k_1 is any bounded analytic solution of $b - k_1 a_0 = \theta h$ for some $h \in H^2$. ∎

Assume now that the inner function θ (as used in this section) has some Blaschke factor, i.e., is zero at some $\alpha \in \mathbb{D}$, and that $|a(\alpha)| = |b(\alpha)|$. If T_φ is hyponormal, i.e.,

$$b - k_1 a = \theta h$$

has a solution with $\|k_1\|_\infty \le 1$, we then have

$$b(\alpha) - k_1(\alpha)\, a(\alpha) = \theta(\alpha)\, h(\alpha) = 0,$$

or $k_1(\alpha) = \frac{b(\alpha)}{a(\alpha)}$ will have modulus one, which means that $k_1(z)$ must be constant, by the maximum modulus principle, say $k_1(z) = \zeta$. We now write

$$b - \zeta(a_0 + \theta a_1) = \theta h,$$

and deduce that since $b - \zeta a_0 \in \mathcal{H}(\theta)$, $b - \zeta a_0 = 0$. This then gives us

Corollary 8. *If $\varphi = \theta_0\theta\overline{a} + \overline{\theta}b$, for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$, and $|a(\alpha)| = |b(\alpha)|$ for some α with $\theta(\alpha) = 0$, then T_φ is hyponormal iff $b = \zeta a_0$ for some constant ζ of modulus 1.*

6.1. **Examples.** Consider again the case where φ is any trigonometric polynomial, $\varphi(z) = \sum_{n=-m}^{N} c_n z^n$, with $N \geq m$. We can write $\varphi = \theta_0\theta\overline{a} + c_0 + \overline{\theta}b$ where $\theta(z) = z^m$, $\theta_0(z) = z^{N-m}$, $a(z) = c_N + c_{N-1}z + \cdots + c_1 z^{N-1}$, and $b(z) = c_{-1}z^{m-1} + c_{-2}z^{m-2} + \cdots + c_{-m}$. When we write $a = P_{\mathcal{H}(\theta)}a + \theta a_1 = a_0 + \theta a_1$ as above, we see that $a_0(z) = c_N + c_{N-1}z + \cdots + c_{N-m+1}z^{m-1}$, and $a_1(z) = c_{N-m}z^m + c_{N-m-1}z^{m+1} + \cdots + c_1 z^{N-1}$. We thus see that the hyponormality of T_φ is independent of c_0, and, since it is independent of a_1, it is independent of the Fourier coefficients $c_1, \ldots, c_{N-m}$ as well. This is a result demonstrated in a different way in [6, Theorem 1].

Furthermore T_φ is hyponormal if and only if T_ψ is hyponormal where $\psi(z) = \overline{T_{\overline{z^{N-m}}} z^N \overline{a} + \overline{z^m} b} = z^m \overline{a_0(z)} + \overline{z}^m b$, which is obtained in [15, Proposition 2]. This result shows that to determine the hyponormality of T_φ with $\varphi(z)$ of the form $\varphi(z) = \sum_{n=-m}^{N} c_n z^n$ $(m \leq N)$, it is sufficient to consider the Toeplitz operator T_ψ with $\psi(z)$ of the form $\psi(z) = \sum_{n=-m}^{m} c_n z^n$.

Also, since $\theta(z) = z^m$, if $|a(0)| = |b(0)|$, i.e., $|c_N| = |c_{-m}|$, and T_φ is hyponormal, Corollary 8 says that we must have some constant ζ of modulus one with $b = \zeta a_0$, i.e., $c_{-m+k} = \zeta c_{N-k}$ for $k = 0, 1, \ldots m-1$. This is a result which was shown in [5, Theorem 1.4].

7. Norms of Hankel operators

In this section we give more explicit examples of hyponormal Toeplitz operators with symbols satisfying some symmetric conditions including polynomial symbols discussed in [5] and [15]. We first introduce some notation. Let $\varphi = \theta_0\theta\overline{a} + \overline{\theta}b$, for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$. Let also $\theta = \theta_1\theta_2$ where θ_2 is a finite Blaschke product. Assume

$$b = \beta a + \theta_1 b_1 \text{ for some } b \in \mathcal{H}(\theta_2) \text{ and constant } \beta\ (|\beta| \leq 1). \tag{7.1}$$

Note that in this case equation (6.1) becomes

$$\beta a + \theta_1 b_1 - k_1 a = \theta_1\theta_2 h.$$

This implies that $(\beta - k_1)a = \theta_1 h_1$ for some $h_1 \in H^2$. Therefore $k_1 = \beta + \theta_1 k_2$ where k_2 satisfies

$$b_1 - k_2 a = \theta_2 h_2 \tag{7.2}$$

for some $h_2 \in H^2$. By Theorem 8, T_φ, with φ satisfying (7.1), is hyponormal if and only if $\left\| H_{\overline{\theta_1\theta_2}(\beta+\theta_1 k_2)} \right\| \leq 1$.

The efficient computation of the norms of more general Hankel operators, of the form $\left\| H_{\overline{\theta_1\theta_2}(\beta+\theta_1 k_2)} \right\|$, where β is a rational function (instead of a constant) and k_2 is any function in H^∞, has been studied extensively in recent literature because of its important application in robust control theory. We refer reader to the book [9] by Foias, Özbay and Tannenbaum for details. In particular, if $\gamma = \left\| H_{\overline{\theta_1\theta_2}(\beta+\theta_1 k_2)} \right\| > \gamma_e$, where $\gamma_e = \left\| H_{\overline{\theta_1\theta_2}(\beta+\theta_1 k_2)} \right\|_e$ is the essential norm of $H_{\overline{\theta_1\theta_2}(\beta+\theta_1 k_2)}$, then there exists an efficiently computable matrix $K(\gamma)$ such that γ is the largest value for which $K(\gamma)$ is singular. The remarkable fact is that $K(\gamma)$ is of order at most $2(n+l)$ where n is the degree of β and l is the degree of the finite Blaschke product θ_2. There are several efficient algorithms to compute $K(\gamma)$; see the paper [12] by the first named author, Toker and Özbay for one algorithm which uses the realization form of the rational functions β and θ_2. Furthermore γ_e can be computed as shown in Foias and Tannenbaum [8].

We next give examples by further assuming $\theta_2 = z$. Thus we can take k_2 satisfying (7.2) to be a constant. In other words, we will compute $\gamma = \left\| H_{\overline{\theta}\overline{z}(1+\alpha\theta)} \right\|$ where α is any constant. Since the results in [12] are derived for Hankel operators on the Hardy space of the right half plane for the much more general case, here we give a self-contained discussion for the computation of $\left\| H_{\overline{\theta}\overline{z}(1+\alpha\theta)} \right\|$.

Let $\gamma_e = \left\| H_{\overline{\theta}\overline{z}(1+\alpha\theta)} \right\|_e$ be the essential norm of $H_{\overline{\theta}\overline{z}(1+\alpha\theta)}$. First note that since $H_{\overline{\theta}\overline{z}(1+\alpha\theta)} = H_{\overline{\theta}\overline{z}} + H_{\alpha\overline{z}}$, $H_{\overline{\theta}\overline{z}(1+\alpha\theta)}$ is a rank one perturbation of $H_{\overline{\theta}\overline{z}}$, we know that $H_{\overline{\theta}\overline{z}(1+\alpha\theta)}$ and $H_{\overline{\theta}\overline{z}}$ have the same essential norm. If θ is a finite Blaschke product, then $H_{\overline{\theta}\overline{z}(1+\alpha\theta)}$ is of finite rank, so $\gamma_e = 0$. If θ has as a factor an infinite Blaschke product or a singular inner function, then $\gamma_e = 1$, since $H^*_{\overline{\theta}\overline{z}} H_{\overline{\theta}\overline{z}}$ is the projection onto the infinite dimensional space $\mathcal{H}(z\theta)$.

Assume now $\gamma = \left\| H_{\overline{\theta}\overline{z}(1+\alpha\theta)} \right\| > \gamma_e$. Then γ is the largest value such that there exists a pair of vectors x and y_0 in H^2 (called a Schmidt pair) satisfying

$$\begin{aligned} H_{\overline{\theta}\overline{z}(1+\alpha\theta)} x &= J(I-P)\left[\overline{\theta}\overline{z}(1+\alpha\theta)x\right] = \gamma y_0, \\ H^*_{\overline{\theta}\overline{z}(1+\alpha\theta)} y_0 &= P\left[\overline{\theta}\overline{z}(1+\alpha\theta)J^* y_0\right] = \gamma x. \end{aligned}$$

Write $J^* y_0 = \overline{zy}$ where $y \in H^2$. The above equation holds if and only if there exist ϕ_1 and ϕ_2 in H^2 such that

$$\overline{\theta}\overline{z}(1+\alpha\theta)x + \phi_1 = \gamma\overline{zy}, \tag{7.3}$$

$$\theta z(1+\overline{\alpha}\overline{\theta})\overline{zy} + \overline{z\phi_2} = \gamma x. \tag{7.4}$$

It follows from above equations that

$$\gamma\overline{\theta}\overline{z}x = (1+\overline{\alpha}\overline{\theta})\overline{zy} + \overline{\theta}\overline{z}^2\overline{\phi_2} \in \overline{zH^2}, \quad \gamma\theta\overline{y} = (1+\alpha\theta)x + \theta z\phi_1 \in H^2.$$

Therefore

$$\phi_1 = -P\left[\overline{\theta}\overline{z}(1+\alpha\theta)x\right] = -P(\overline{\theta}\overline{z}x) - \alpha P(\overline{z}x) = -\alpha\overline{z}\left[x - x(0)\right] \tag{7.5}$$

and

$$\begin{aligned} \overline{z\phi_2} &= -(I-P)\left[\theta z(1+\overline{\alpha}\overline{\theta})\overline{zy}\right] \\ &= -(I-P)\left[\theta\overline{y}\right] - \overline{\alpha}(I-P)\left[\overline{y}\right] = -\overline{\alpha}\left[\overline{y} - \overline{y(0)}\right]. \end{aligned} \tag{7.6}$$

Plugging (7.5) and (7.6) into (7.3) and (7.4), we get

$$\begin{aligned} \overline{\theta}x + \alpha x(0) &= \gamma\overline{y}, \\ \theta\overline{y} + \overline{\alpha}\overline{y(0)} &= \gamma x. \end{aligned}$$

Solving x and $\overline{y}$, we have

$$\begin{aligned}\left(\gamma^2-1\right)\overline{y} &= \overline{\alpha}\overline{\theta}\overline{y(0)}+\gamma\alpha x(0),\\ \left(\gamma^2-1\right)x &= \alpha\theta x(0)+\gamma\overline{\alpha}\overline{y(0)}.\end{aligned}$$

Therefore there exist non-zero solutions x and $\overline{y}$ satisfying the above equations if and only if there exist non-zero numbers $x(0)$ and $\overline{y}(0)$ satisfying

$$\begin{aligned}\left(1+\overline{\alpha}\overline{\theta}(0)-\gamma^2\right)\overline{y(0)}+\gamma\alpha x(0) &= 0\\ \left(1+\alpha\theta(0)-\gamma^2\right)x(0)+\gamma\overline{\alpha}\overline{y(0)} &= 0\end{aligned}$$

This happens if and only if the 2×2 matrix

$$K(\gamma)=\begin{pmatrix} 1+\overline{\alpha}\overline{\theta}(0)-\gamma^2 & \gamma\alpha \\ \gamma\overline{\alpha} & 1+\alpha\theta(0)-\gamma^2 \end{pmatrix}$$

is singular. Equivalently

$$\left|1+\alpha\theta(0)-\gamma^2\right|^2-|\gamma\alpha|^2=0.$$

Solving the above equation, we see that $\gamma=\left\|H_{\overline{\theta}\overline{z}(1+\alpha\theta)}\right\|$ is given by

$$\gamma^2=\frac{2+2\operatorname{Re}[\alpha\theta(0)]+|\alpha|^2+\sqrt{4|\alpha|^2(1+2\operatorname{Re}[\alpha\theta(0)])+|\alpha|^4-4|\operatorname{Im}[\alpha\theta(0)]|^2}}{2}$$

Theorem 9. *Let $\varphi=z\theta\overline{a}+\overline{z}\left(\overline{\theta}\beta a+a(0)\delta\right)$, for $a\in\mathcal{H}(z\theta)$ and some constants δ and β ($|\beta|\le 1$). Then T_φ is hyponormal if and only if $\left\|H_{\overline{\theta}\overline{z}(\beta+\delta\theta)}\right\|\le 1$, or for $\alpha=\delta/\beta$,*

$$\frac{2+2\operatorname{Re}[\alpha\theta(0)]+|\alpha|^2+\sqrt{4|\alpha|^2(1+2\operatorname{Re}[\alpha\theta(0)])+|\alpha|^4-4|\operatorname{Im}[\alpha\theta(0)]|^2}}{2}\le\frac{1}{|\beta|^2}.$$

Proof. If we let $b=\beta a+a(0)\delta\theta$, equation (7.2) becomes $a(0)\delta-k_2a=zh_2$ for some $h_2\in H^2$. Thus we can take $k_2=\delta$. Therefore T_φ is hyponormal if and only if $\left\|H_{\overline{\theta}\overline{z}(\beta+\delta\theta)}\right\|\le 1$ or $\left\|H_{\overline{\theta}\overline{z}(1+\alpha\theta)}\right\|\le 1/|\beta|$ for $\alpha=\delta/\beta$. The result follows now from above discussion. ∎

7.1. **Example.** Consider the case where φ is any trigonometric polynomial, $\varphi(z)=\sum_{n=-N}^{N}c_nz^n$. We can write $\varphi=z\theta\overline{a}+c_0+\overline{z}\overline{\theta}b$ where $\theta(z)=z^{N-1}$, $a(z)=c_N+c_{N-1}z+\cdots+c_1z^{N-1}$, and $b(z)=c_{-1}z^{N-1}+c_{-2}z^{N-2}+\cdots+c_{-N}$. Assume $c_{-k}=\beta c_k$ for $k=2,\cdots N$. That is $b(z)=\beta a(z)+c_N\delta\theta(z)$, for $\delta=(c_{-1}-\beta c_1)/c_N$. T_φ is hyponormal if and only if

$$2+\left|\frac{\delta}{\beta}\right|^2+\sqrt{4\left|\frac{\delta}{\beta}\right|^2+\left|\frac{\delta}{\beta}\right|^4}\le\frac{2}{|\beta|^2},$$

or $|\delta|\le 1-|\beta|^2$. Note that $\beta=c_{-N}/c_N$. So T_φ is hyponormal if and only if

$$|c_{-1}c_N-c_1c_{-N}|\le|c_N|^2-|c_{-N}|^2.$$

The sufficiency of this condition is obtained in [5, Theorem 1.8] while the necessity of this condition is proved in [15, Theorem 6].

We conclude the paper with some explicit examples of hyponormal Toeplitz operators with irrational symbols. Let

$$\varphi(z) = ze^{-\frac{1+z}{1-z}} + \beta\overline{z}e^{-\frac{1+\overline{z}}{1-\overline{z}}} + \beta\overline{z}$$

($\theta = e^{-\frac{1+z}{1-z}}, a = 1, b = \beta + \beta\theta$, in the notation of Theorem 9). T_φ is hyponormal if and only if

$$|\beta|^2 \leq \frac{2}{3 + 2e^{-1} + \sqrt{5 + 8e^{-1}}} \approx .305\,15.$$

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California Polytechnic State University, San Luis Obispo
E-mail address: cgu@calpoly.edu

E-mail address: jshapiro@calpoly.edu
URL: http://www.calpoly.edu/~jshapiro