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Integral Equations and Operator Theory

Continuity of the Norm of a Composition Operator

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Abstract. We explore the continuity of the map which, given an analytic self-map of the disk, takes as its value the norm of the associated composition operator on the Hardy space H^2. We also examine the continuity the functions which assign to a self-map of the disk the Hilbert-Schmidt norm or the essential norm of the associated composition operator and show these to be discontinuous. Additionally, we characterize when the norm of a composition operator is minimal.

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1. Introduction

Let $\mathbb{D}$ denote the unit disc of the complex plane and φ be a holomorphic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The equation $C_\varphi f = f \circ \varphi$ defines a *composition operator* on the Hardy Space H^2. For any such φ, C_φ is a bounded operator (see [4, pg. 117]). It is known that if $\varphi_n \to \varphi$ weakly in H^1 then $C_{\varphi_n} \to C_\varphi$ weakly. If $\varphi_n \to \varphi$ in H^1 then $C_{\varphi_n} \to C_\varphi$ pointwise or strongly. These results and others are found in [7]. It is also known that $\varphi_n \to \varphi$ in some sense does not in general imply that $\|C_\varphi - C_{\varphi_n}\| \to 0$. However, we show that in several cases, it does imply that $\|C_{\varphi_n}\| \to \|C_\varphi\|$.

Let $ASM(\mathbb{D})$ be the set of all analytic self-maps of the disk, considered as a subset of H^1 or H^∞. We will denote this set $ASM(\mathbb{D})^1$ or $ASM(\mathbb{D})^\infty$ respectively, when we wish to make a distinction. Recall that the H^∞ norm is given by $\|\varphi\|_\infty =$

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$\sup_{|z|<1} |\varphi(z)|$, and the H_1 norm is given by

$$\|\varphi\|_1 = \int_0^{2\pi} |\varphi(e^{i\theta})| \, \frac{d\theta}{2\pi}.$$

Let $N\colon ASM(\mathbb{D}) \to \mathbf{R}$ be given by $N(\varphi) = \|C_\varphi\|$. We will use $N_1\colon ASM(\mathbb{D})^1 \to \mathbf{R}$ and $N_\infty\colon ASM(\mathbb{D})^\infty \to \mathbf{R}$ to distinguish the topologies on the domain ($N \equiv N_1 \equiv N_\infty$). Stated in a different way, we are asking:

For which functions $\varphi \in ASM(\mathbb{D})$ *is* N_1 *or* N_∞ *continuous?*

In addition to questions of the convergence of $\|C_{\varphi_n}\|$ to $\|C_\varphi\|$, we will consider similar questions with the operator norm replaced with either the essential norm $\| \, \|_e$ or the Hilbert-Schmidt norm $\| \, \|_{HS}$. Recall that $\|T\|_e$ is the distance from T to the subspace of compact operators in the operator norm topology and

$$\|T\|_{HS}^2 = \sum_{n=1}^\infty \|Te_n\|^2$$

where $\{e_n\}_1^\infty$ is a complete orthonormal basis. We will also often suppress the subscript and write $\|f\| = \|f\|_2$ for the H^2 norm of an analytic function in $\mathbb{D}$.

In Section 5, we show that the well known lower bound for the norm of a composition operator C_φ, equation (3.1), is attained (when $\varphi(0) \neq 0$) only when φ is a constant function. This appears here as Theorem 4. Chris Hammond, in [6], has recently obtained this same theorem with a different proof.

2. Hilbert-Schmidt Operators

Let $ASM(\mathbb{D})_{HS}^\infty$ denote the set of analytic self-maps of $\mathbb{D}$ that induce Hilbert-Schmidt composition operators, and give this set the H^∞ norm. Suppose that $\varphi_n \to \varphi$ are all in $ASM(\mathbb{D})_{HS}^\infty$. Does it then follow that $\|C_{\varphi_n}\|_{HS} \to \|C_\varphi\|_{HS}$? In general, the answer is no, as the example below demonstrates. Let

$$\varphi(z) = \left[1 + \left(\frac{1-z}{1+z}\right)^{1/2}\right]^{-1}$$

and

$$\varphi_n(z) = \left[1 - \left(\frac{1-\frac{n}{n+1}}{2}\right)^{1/2} + \left(\frac{1-\frac{n}{n+1}z}{1+z}\right)^{1/2}\right]^{-1}$$

where the real part of $z^{1/2}$ is always positive.

We approach the Hilbert-Schmidt norms of composition operators by using the formulas (as in [4, Sec. 3.3])

$$\langle C_\varphi, C_\psi \rangle_{HS} = \int_0^{2\pi} \frac{1}{1-\varphi(e^{i\theta})\overline{\psi(e^{i\theta})}} \, \frac{d\theta}{2\pi}$$

and

$$\|C_\varphi\|_{HS}^2 = \int_0^{2\pi} \frac{1}{1-|\varphi(e^{i\theta})|^2}\,\frac{d\theta}{2\pi}.$$

To prove that C_φ is Hilbert-Schmidt, we will show by direct computation that the integral

$$\int_0^{\pi} \frac{1}{1-|\varphi(e^{i\theta})|^2}\,\frac{d\theta}{2\pi}$$

is finite. Since $\varphi(e^{-i\theta}) = \overline{\varphi(e^{i\theta})}$ this will show that $\|C_\varphi\|_{HS}$ is finite. First note that for $0 < \theta < \pi$,

$$\frac{1-e^{i\theta}}{1+e^{i\theta}} = \frac{\tan(\theta/2)}{i},$$

and if we set $t = \tan(\theta/2)$,

$$\varphi(e^{i\theta}) = \frac{1}{1+\sqrt{\frac{t}{i}}} = \frac{1}{1+\frac{\sqrt{t}}{\sqrt{2}} - i\frac{\sqrt{t}}{\sqrt{2}}}$$

so

$$|\varphi(e^{i\theta})|^2 = \frac{1}{1+\sqrt{2t}+t}$$

and thus

$$\frac{1}{1-|\varphi(e^{i\theta})|^2} = \frac{1}{1-\frac{1}{1+\sqrt{2t}+t}} = 1 + \frac{1}{\sqrt{2t}+t}.$$

It can be shown using basic calculus that the integral

$$\int_0^{\pi} \left(1 + \frac{1}{\sqrt{2\tan(\theta/2)}+\tan(\theta/2)}\right)\frac{d\theta}{2\pi}$$

converges. Each of the maps φ_n has an angular derivative at 1 because each φ_n is in fact analytic in a disc centered at 1. Thus none of the C_{φ_n} are compact and thus none have finite Hilbert-Schmidt norm.

For each n, choose $\frac{n}{n+1} < r_n < 1$ such that $\|C_{r_n\varphi_n}\|_{HS} > n$. Note that each $r_n\varphi_n$ induces a Hilbert-Schmidt operator. Now $\|C_{r_n\varphi_n}\|_{HS} \to \infty$ while $r_n\varphi_n \to \varphi$ in H^∞.

What about φ such that $\|\varphi\|_\infty < 1$? Is the map $N_{HS}(\varphi) = \|\varphi\|_{HS}$ continuous at such points? Suppose that $\|\varphi\|_\infty < 1$ and $\varphi_n \to \varphi$ in $ASM(\mathbb{D})^\infty_{HS}$. There exists an $r < 1$ such that for sufficiently large n, $\|\varphi_n\|_\infty < r$. Thus

$$\begin{aligned}\|C_{\varphi_n} - C_\varphi\|_{HS}^2 &= \|C_\varphi\|_{HS}^2 + \|C_{\varphi_n}\|_{HS}^2 - \langle C_\varphi, C_{\varphi_n}\rangle_{HS} - \langle C_{\varphi_n}, C_\varphi\rangle_{HS} \\ &= \int_0^{2\pi}\frac{1}{1-|\varphi_n|^2} + \int_0^{2\pi}\frac{1}{1-|\varphi|^2} - \int_0^{2\pi}\frac{1}{1-\varphi_n\overline{\varphi}} - \int_0^{2\pi}\frac{1}{1-\overline{\varphi}_n\varphi}.\end{aligned}$$

This in turn gives

$$\|C_{\varphi_n} - C_\varphi\|_{HS}^2 = \int_0^{2\pi}\frac{\overline{\varphi}(\varphi-\varphi_n)}{(1-|\varphi|^2)(1-\varphi_n\overline{\varphi})} + \int_0^{2\pi}\frac{\overline{\varphi}_n(\varphi_n-\varphi)}{(1-|\varphi_n|^2)(1-\overline{\varphi}_n\varphi)}$$

$$\leq \frac{2}{(1-r^2)^2}\int_0^{2\pi}|\varphi_n-\varphi| \to 0$$

as $n \to \infty$. It now follows that $\|C_{\varphi_n}\|_{HS} \to \|C_\varphi\|_{HS}$. We thus have the following

Theorem 1. *N_{HS} is continuous at all φ with $\|\varphi\|_\infty < 1$.*

Does this theorem hold if we replace the norm on the space $ASM(\mathbb{D})^\infty_{HS}$ with the H^1 norm? No: Let

$$\eta(z) = 1 - \frac{2}{1+\left(i(\frac{2}{1-z}-1)\right)^{1/2}}$$

and

$$\alpha_p(z) = \frac{p-z}{1-\overline{p}z} \qquad (p \in \mathbb{D}).$$

Then $\eta(z)$ maps $\mathbb{D}$ to the upper half-disk (with $-i$ mapping to 0) and α_p is the automorphism of the disk that sends 0 to p. For any $p \in \mathbb{D}$, $\|C_{\eta\circ\alpha_p}\|_{HS} = \infty$ because $|\eta \circ \alpha_p(z)| = 1$ on a set of positive measure (an arc of the upper half-circle in a neighborhood of i). For each n, let $\eta_n = \eta \circ \alpha_{-i(1-\frac{1}{n})}$. Then, as n gets large, η_n maps the unit circle (at least all of it away from some neighborhood of i) to arbitrarily small segments on the real line centered at 0, so $\|\eta_n\|_1 \to 0$ as $n \to \infty$. Choose r_n such that $\frac{n}{n+1} < r_n < 1$ and $\|C_{r_n\eta_n}\|_{HS} > n$. We now have $\|C_{r_n\eta_n}\|_{HS} \to \infty$ and $r_n\eta_n \to 0$ in H^1 as $n \to \infty$.

Remark: It is easy to check that if $\varphi \in ASM(\mathbb{D})$ satisfies $\|C_\varphi\|_{HS} = \infty$ then $\|C_{r\varphi}\|_{HS}$ is unbounded for $r \in (0,1)$.

3. Continuity of the norm

We now address the first question raised in the introduction. Is N_1 or N_∞ continuous? As the H^∞ topology is stronger than H^1 topology, a map φ at which N_1 is continuous is a map at which N_∞ is continuous.

Theorem 2. *The map N_1 (and thus N_∞) is continuous at φ if either $\varphi(0) = 0$ or φ is an inner map. The map N_∞ is continuous at φ which satisfy $\varphi(\mathbb{D}) \subseteq r\mathbb{D}$ for some $r < 1$.*

Proof. The following estimate for the norm of a composition operator is well-known (for instance in [4, Cor. 3.7]):

$$\frac{1}{\sqrt{1-|\varphi(0)|^2}} \leq \|C_\varphi\| \leq \frac{1+|\varphi(0)|}{\sqrt{1-|\varphi(0)|^2}} \tag{3.1}$$

Furthermore, it is noted in [8] that the upper bound is attained if φ is inner. If $\varphi(0) = 0$, then $\|C_\varphi\| = 1$ and $\varphi_n(0) \to 0$, so the bounds given above force $\lim_{n\to\infty}\|C_{\varphi_n}\| = 1$. Let $T \in \mathcal{L}(H^2)$ be a bounded linear operator. Define $\|T\|_{(m)} =$

$\|TK_m\|$ where K_m is the orthogonal projection of H^2 onto the subspace spanned by $\{1, z, z^2, \ldots, z^{m-1}\}$. Then

$$\|T\|_{(n)} \le \|T(1)\| + \|T(z)\| + \cdots + \|T(z^{m-1})\|,$$

so if $\varphi_n \to \varphi$ in H^1 (and thus in H^2), one obtains

$$\|C_{\varphi_n} - C_\varphi\|_{(n)} \le \|\varphi_n - \varphi\|_2 + \|\varphi_n^2 - \varphi^2\|_2 + \cdots + \|\varphi_n^{m-1} - \varphi^{m-1}\|_2.$$

Since

$$\|\varphi_n^k - \varphi^k\|_2 \le \|\varphi_n - \varphi\|_2 \|\varphi_n^{k-1} + \cdots + \varphi^{k-1}\|_\infty \le k\|\varphi_n - \varphi\|_2,$$

we have $\|C_{\varphi_n} - C_\varphi\|_{(m)} \le m^2 \|\varphi_n - \varphi\|_2$. Thus $\varphi \mapsto \|C_\varphi\|_{(m)}$ is a continuous map on $ASM(\mathbb{D})^1$ for all m. As $\lim_{m\to\infty} \|C_\varphi\|_{(m)} = \|C_\varphi\|$ is an increasing limit for all $\varphi \in ASM(\mathbb{D})^1$, $N_1(\varphi)$ is lower semi-continuous for all φ. In other words, if φ is given and $\varepsilon > 0$ then there exists $\delta > 0$ such that $\|\varphi - \psi\| \le \delta$ implies $\|C_\psi\| \ge \|C_\varphi\| - \varepsilon$. Note that if $\varphi_n \to \varphi$ in H^1 then $\varphi_n(0) \to \varphi(0)$. This can be seen using properties of subharmonic functions (see [5, Th. 2.6 and 1.4]). For each n, $g_n(z) = \varphi_n(z) - \varphi(z)$ is a bounded analytic function in D and $|g_n|$ is subharmonic so

$$\|g_n\|_1 = \lim_{r\to 1^-} \int_0^{2\pi} |g_n(re^{i\theta})| \, \frac{d\theta}{2\pi} \ge |g_n(0)|.$$

If, on the other hand, φ is inner, then

$$\lim_{n\to\infty} \|C_{\varphi_n}\| \le \lim_{n\to\infty} \frac{1 + |\varphi_n(0)|}{\sqrt{1 - |\varphi_n(0)|^2}} = \frac{1 + |\varphi(0)|}{\sqrt{1 - |\varphi(0)|^2}} = \|C_\varphi\|.$$

Since $N_1(\varphi)$ is lower semi-continuous, this gives

$$\lim_{n\to\infty} \|C_{\varphi_n}\| = \|C_\varphi\|.$$

Suppose now that $\varphi_n \to \varphi$ in H^∞ and $\varphi(\mathbb{D}) \subseteq r\mathbb{D}$ for some $r < 1$. In this case, there exists an $r' < 1$ such that $|\varphi_n(z)| < r'$ for sufficiently large n. We have already shown, while proving Theorem 1, that we can conclude that $\|C_{\varphi_n} - C_\varphi\|_{HS}^2 \to 0$, so $\|C_{\varphi_n} - C_\varphi\| \to 0$ and hence $\|C_{\varphi_n}\| \to \|C_\varphi\|$. □

This theorem shows that, in several cases, $\|C_{\varphi_n}\| \to \|C_\varphi\|$ when $\varphi_n \to \varphi$ in H^1. In general, it is not true that $\|C_\varphi - C_{\varphi_n}\| \to 0$. It is well known that $\|C_\varphi - C_{\varphi_n}\|$ can stay large, even when $\varphi_n \to \varphi$ in H^∞. As a simple example, consider the functions $\varphi_n(z) = e^{i\frac{\pi}{n}} z$ and $\varphi(z) = z$. Here $\|(C_\varphi - C_{\varphi_n})(z^n)\| = \|z^n - (e^{i\frac{\pi}{n}} z)^n\| = \|z^n + z^n\| = 2$, so $\|C_{\varphi_n} - C_\varphi\| \ge 2$ for all n.

4. The essential norm

Let $\varphi_n(z) = (1 - \frac{1}{n})z$ and $\varphi(z) = z$. Since $\|\varphi_n\|_\infty = 1 - \frac{1}{n} < 1$, each C_{φ_n} is compact and hence $\|C_{\varphi_n}\|_e = 0$ for all n (see [9] for details). Using the formula for the essential norm given in [10], we have $\|C_\varphi\|_e = \limsup_{|w|\to 1} N_\varphi(w)/(-\log|w|) = 1$ (here N_φ is the Nevanlinna counting function). Thus the map $\varphi \mapsto \|C_\varphi\|_e$ is not continuous everywhere on $ASM(\mathbb{D})^\infty$. In fact we have the following

Theorem 3. *If the map* $\varphi \mapsto \|C_\varphi\|_e$ *on* $ASM(\mathbb{D})^\infty$ *is continuous at* φ*, then* C_φ *is a compact operator.*

Proof. Suppose that C_φ is not a compact operator. Let $\varphi_n = \frac{n}{n+1}\varphi$. We have $\|C_{\varphi_n}\|_e = 0$ for all n, but $\|C_\varphi\|_e > 0$. □

The converse of this theorem remains open. For reference, define the maps $N_{1,e}$ on $ASM(D)^1$ and $N_{\infty,e}$ on $ASM(D)^\infty$ by $N_{1,e}(\varphi) = N_{\infty,e}(\varphi) = \|C_\varphi\|_e$.

5. Composition operators of minimal norm

Consider the upper and lower bounds for the norm of a composition operator on H^2 that appear in equation (3.1). It is known precisely when $\|C_\varphi\|$ achieves this upper bound (see [8]). Namely, if $\varphi(0) = 0$, then $\|C_\varphi\| = 1$, and if $\varphi(0) \neq 0$ then $\|C_\varphi\| = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$ if and only if φ is inner.

Suppose that φ is now a constant function. A straightforward calculation shows that $\|C_\varphi\| = \sqrt{\frac{1}{1-|\varphi(0)|^2}}$. Does this equation characterize the constant analytic self-maps of D? The answer is yes:

Theorem 4. *Suppose that* φ *is an analytic self-map of* D*,* $\varphi(0) \neq 0$*, and* φ *is not constant. Then*

$$\|C_\varphi\| > \sqrt{\frac{1}{1-|\varphi(0)|^2}}.$$

Proof. Let $\varphi(0) = a$ and without loss of generality assume that a is real and positive. Since non-constant analytic maps are open, there is a real $b > a$ such that $\varphi(z) = b$ for some $z \in D$. Let $x > 0$ and observe that

$$\frac{\|C_\varphi^*(K_0 + xK_z)\|}{\|K_0 + xK_z\|} \leq \|C_\varphi\|$$

where $K_t(w) = \frac{1}{1-\bar{t}w}$ is the reproducing kernel at $t \in D$. We have

$$\begin{aligned}\frac{\|C_\varphi^*(K_0 + xK_z)\|^2}{\|K_0 + xK_z\|^2} &= \frac{\frac{1}{1-a^2} + \frac{2x}{1-ab} + \frac{x^2}{1-b^2}}{1 + 2x + \frac{x^2}{1-|z|^2}} \\ &= \frac{1}{1-a^2}\left(\frac{1 + 2x\frac{1-a^2}{1-ab} + x^2\frac{1-a^2}{1-b^2}}{1 + 2x + \frac{x^2}{1-|z|^2}}\right).\end{aligned}$$

Now notice that since $b > a > 0$, we have $\frac{1-a^2}{1-ab} > 1$. If x satisfies $0 < x < 2(1-|z|^2)\left(\frac{1-a^2}{1-ab} - 1\right)$, then

$$\frac{x^2}{1-|z|^2} < 2x\frac{1-a^2}{1-ab} - 2x,$$

so

$$1 + 2x + \frac{x^2}{1-|z|^2} < 1 + 2x\frac{1-a^2}{1-ab} + x^2\frac{1-a^2}{1-b^2}$$

and

$$1 < \frac{1 + 2x\frac{1-a^2}{1-ab} + x^2\frac{1-a^2}{1-b^2}}{1 + 2x + \frac{x^2}{1-|z|^2}}.$$

We thus have

$$\frac{1}{1-a^2} < \frac{\|C_\varphi^*(K_0 + xK_z)\|^2}{\|K_0 + xK_z\|^2} \leq \|C_\varphi\|^2.$$

☐

6. Conclusion

We have examined the continuity of several types of norms on composition operators with respect the symbols of those composition operators. We have proved that the map N_1 is lower semi-continuous and, indeed, continuous at all φ which either map 0 to 0, or are inner, and N_∞ is continuous at all φ which map $\mathbb{D}$ into $r\mathbb{D}$ for some $r < 1$.

To our knowledge, other functions such as $N_{L_2}(\varphi) = \sup_{w \in D} \frac{\|K_{\varphi(w)}\|}{\|K_w\|}$ and $N_{L_3}(\varphi) = \sup_{w \in D} \frac{\|C_\varphi K_w\|}{\|K_w\|}$ have not yet been investigated in terms of continuity. (The L_* notation follows [2].) The continuity of these maps could possibly have some relationship with that of N_1, since, for φ in a large subset of $ASM(\mathbb{D})$, we have

$$\|C_\varphi\| = \sup_{w \in D} \frac{\|K_{\varphi(w)}\|}{\|K_w\|} = \sup_{w \in D} \frac{\|C_\varphi K_w\|}{\|K_w\|}.$$

This is not true for all φ in $ASM(\mathbb{D})$, as was shown in [1] and discussed further in [2] and [3].

We conclude with the following conjecture:

Conjecture 1. *The map $N_1(\varphi)$ is a continuous function on $ASM(\mathbb{D})^1$.*

This would in turn demonstrate the continuity of N_∞.

We also ask if $N_{1,e}$ and $N_{\infty,e}$ are continuous at φ if and only if C_φ is a compact operator.

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