ADJOINTS OF COMPOSITION OPERATORS WITH IRRATIONAL SYMBOL

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ABSTRACT. In this paper we derive formulas for the adjoints of a class of composition operators with irrational symbol, in particular, the \( n \)-th root functions. We discuss these formulas on both the Hardy space and the Bergman space.

1. INTRODUCTION

Let \( \mathbb{D} \) be the open unit disk of the complex plane \( \mathbb{C} \). Let \( H(\mathbb{D}) \) be the space of all analytic functions on \( \mathbb{D} \). If \( \varphi \) is an analytic self-map of \( \mathbb{D} \), then the composition operator \( C_\varphi \) is defined on \( H(\mathbb{D}) \) by \( C_\varphi f = f \circ \varphi \) for \( f \in H(\mathbb{D}) \). The operators \( C_\varphi \) on subspaces of \( H(\mathbb{D}) \) such as the Hardy space \( H^2(\mathbb{D}) \) and the Bergman space \( A^2(\mathbb{D}) \) have been studied intensively for several decades [8, 22]. One of the useful tools to study \( C_\varphi \) is to find an explicit formula for the adjoint \( C_\varphi^* \). The formula for \( C_\varphi^* \) has proven to be of central importance in dealing with questions of norm computations of \( C_\varphi \), self-adjointness, essential normality of \( C_\varphi \), understanding the \( C^* \) algebra generated by \( C_\varphi \), and so on, see [2, 4, 3, 10, 18].

Here is a brief account of what is known about explicit formulas for \( C_\varphi^* \). In 1988, Cowen [6] derived the formula for \( C_\varphi^* \) on the Hardy space \( H^2(\mathbb{D}) \) when \( \varphi \) is a linear fractional transformation. This formula expresses \( C_\varphi^* \) as the product of two Toeplitz operators and another composition operator. This formula was extended to one on the Bergman space \( A^2(\mathbb{D}) \) by Hurst [17] in 1997, again for a linear fractional symbol \( \varphi \). In 2003, Gallardo-Gutiérrez and Montes-Rodríguez [12] obtained the formula for \( C_\varphi^* \) for a linear fractional symbol \( \varphi \) on the Dirichlet space \( D^2(\mathbb{D}) \). The space \( D^2(\mathbb{D}) \) consists of functions in \( H(\mathbb{D}) \) whose derivatives belong to \( A^2(\mathbb{D}) \). It took the effort of several authors [7, 15, 5] to discover and then refine what is now known as the HMR formula for \( C_\varphi^* \) on \( H^2(\mathbb{D}) \) when \( \varphi \) is a rational function. The special case where \( \varphi \) is an inner function was discussed earlier in [20]. Recently Heller [16] also found a formula for \( C_\varphi^* \) on the space \( S^2(\mathbb{D}) \) for a linear fractional symbol \( \varphi \); here \( S^2(\mathbb{D}) \) consists of functions whose derivatives belong to \( H^2(\mathbb{D}) \). There are some extensions of these results to composition operators on holomorphic spaces of several variables, for example, see Cowen and MacCluer [9]. The formula for \( C_\varphi^* \) is also studied on the Hardy space of the half plane by Elliot [11].

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In this paper, we present formulas for $C^*_\varphi$ on $H^2(\mathbb{D})$ or $A^2(\mathbb{D})$ where $\varphi$ is an irrational function in the form $\sqrt[\varphi]{\psi(z)}$ or more generally $\omega\left(\sqrt[\varphi]{\psi(z)}\right)$ for rational functions $\psi(z)$ and $\omega(z)$. In Section 2, we will deal with the case $\varphi = \sqrt[\varphi]{\psi(z)}$ with $\psi(z)$ being a linear fractional map. In Section 3, we illustrate how to extend our idea to the most general case $\varphi = \omega\left(\sqrt[\varphi]{\psi(z)}\right)$ for rational functions $\psi(z)$ and $\omega(z)$.

2. The $n$-th root functions

Throughout the paper, let
\begin{equation}
L(z) = \frac{az + b}{cz + d}
\end{equation}
be a linear fractional transformation.

The following lemma, due to Cowen [6], is used in the formula for the adjoint of $C_{L(z)}$ on $H^2(\mathbb{D})$.

**Lemma 2.1.** The associated linear fractional transformation is defined by
\[ L^*(z) = \frac{1}{L^{-1}(1/z)} = \frac{\pi z - \pi}{-bz + d}. \]

Then $L(z)$ is a self-map of the disk if and only if $L^*(z)$ is also a self-map of the disk.

The following formula can be proved by a direct computation.

**Lemma 2.2.** For $\alpha \in \mathbb{D}$,
\begin{equation}
\frac{1}{1 - \alpha L(z)} = \frac{cz + d}{d - b\alpha} \frac{1}{1 - L^*(\alpha)z}.
\end{equation}

If $L(z)$ is a nonvanishing self-map of the disk (i.e., $L(z) \neq 0$ for $z \in \mathbb{D}$), then $\varphi(z) = \sqrt[n]{L(z)}$ is an analytic self-map of $\mathbb{D}$ by choosing a branch of the logarithm. Recall
\[ H^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\}, \]
and the reproducing kernel of $H^2(\mathbb{D})$ is
\[ K_\alpha(z) = \frac{1}{1 - \alpha z}. \]

For $f(z) \in L^\infty(\mathbb{T})$, the space of bounded functions on the unit circle $\mathbb{T}$, the Toeplitz operator $T_f$ on $H^2(\mathbb{D})$ (or $A^2(\mathbb{D})$) is defined by
\[ T_f : h \mapsto P[f(z)h(z)] = h \in H^2, \]
where $P$ is the projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{D})$ (or the projection from $L^2(\mathbb{D})$ to $A^2(\mathbb{D})$).
Theorem 2.3. Let \( \varphi(z) = \sqrt[2n]{L(z)} \) be a self-map of \( \mathbb{D} \), where \( L \) is a nonvanishing linear-fractional self-map of \( \mathbb{D} \). Then on the Hardy space \( H^2(\mathbb{D}) \),

\[
C_{\varphi(z)}^* = \sum_{k=0}^{n-1} T_{c(z)+d} C_{\varphi^{k}} T_{c(z)+d}^*(z),
\]

where \( \varphi(z) \) is a linear-fractional self-map of \( \mathbb{D} \) by Lemma 2.1, so the composition operator \( C_{L^*}(z^n) \) is a bounded operator. The Toeplitz operators appearing in (2.3) are all bounded.
This approach can be generalized to the reproducing kernel Hilbert space with reproducing kernel

\[
\frac{1}{(1 - \overline{\alpha}z)^m}, \text{ where } m > 0.
\]

The formula for \( C^*_L(z) \) on these spaces was derived by Hurst [17] for \( m > 1 \). When \( m \) is an integer, these spaces are important in operator theory, since the adjoint of multiplication by \( z \) on the vector-valued version of these spaces is a model for \( m \)-hypercontractions \([1, 14]\). We will work out the formula for \( C^*_\phi(z) \) for \( m = 2 \), the Bergman space \( A^2(\mathbb{D}) \). The results for other integers \( m \) are similar. When \( m \) is not an integer, however, an infinite sum is needed. Recall that

\[
A^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\},
\]

where \( dA(z) \) is the normalized Lebesgue area measure of \( \mathbb{D} \). The reproducing kernel of \( A^2(\mathbb{D}) \) is

\[
A_\alpha(z) = \frac{1}{(1 - \overline{\alpha}z)^2}.
\]

**Proposition 1.** Let \( \phi(z) = \sqrt{L(z)} \) be a self-map of \( \mathbb{D} \), where \( L \) is a nonvanishing linear-fractional self-map of \( \mathbb{D} \). Then on the Bergman space \( A^2(\mathbb{D}) \),

\[
C^*_\phi(z) = \sum_{k=0}^{n-1} T_{(k+1)z^k} \frac{(z\overline{\phi})^k}{\alpha^k z^k} T^*_\phi(z^k) + \sum_{k=n}^{2n-2} T_{(2n-k-1)z^k} \frac{(z\overline{\phi})^k}{\alpha^k z^k} T^*_\phi(z^k).
\]

**Proof.** The proof is similar to the Hardy space case. For clarity, we include some details. Note that

\[
A_\alpha(z) = \left( \frac{1}{1 - \overline{\alpha}z} \right)^2 = \left( \sum_{k=0}^{n-1} \frac{\alpha^k z^k}{1 - \overline{\alpha}z} \right)^2
\]

\[
= \frac{1}{(1 - \overline{\alpha}z)^2} \left( \sum_{k=0}^{n-1} (k+1)\alpha^k z^k + \sum_{k=n}^{2n-2} (2n-k-1)\alpha^k z^k \right).
\]

Note also

\[
C^*_\phi f(\alpha) = \langle C^*_\phi f(z), A_\alpha(z) \rangle = (f(z), A_\alpha(\phi(z)))
\]

where the inner product \( \langle f, A_\alpha(\phi(z)) \rangle \) is on \( A^2(\mathbb{D}) \). Therefore

\[
C^*_\phi f(\alpha) = \left\langle f(z), \frac{1}{(1 - \overline{\alpha}L(z))^2} \left( \sum_{k=0}^{n-1} (k+1)\alpha^k z^k + \sum_{k=n}^{2n-2} (2n-k-1)\alpha^k z^k \right) \right\rangle
\]

\[
= \sum_{k=0}^{n-1} \frac{(k+1)\alpha^k}{(d - b\alpha)^2} \left\langle P \left[ (cz + d)^2 \phi f(z) \right], \frac{1}{(1 - L^*(\alpha^n))z} \right\rangle
\]

\[
+ \sum_{k=n}^{2n-2} \frac{(2n-k-1)\alpha^k}{(d - b\alpha)^2} \left\langle P \left[ (cz + d)^2 \phi f(z) \right], \frac{1}{(1 - L^*(\alpha^n))z} \right\rangle
\]

\[
= \sum_{k=0}^{n-1} \frac{(k+1)\alpha^k}{(d - b\alpha)^2} g_k(L^*(\alpha^n)) + \sum_{k=n}^{2n-2} \frac{(2n-k-1)\alpha^k}{(d - b\alpha)^2} g_k(L^*(\alpha^n)).
\]
where the fourth equality is by (2.2), and

\[ g_k(z) = P \left[ (cz + d)^2 \varphi^k f(z) \right] = T^*_k (cz + d)^2 \varphi^k f, \quad 0 \leq k \leq 2n - 2. \]

The formula (2.6) follows from the above calculation. 

**Example 2.6.** Let \( \varphi(z) = \sqrt{\frac{1}{1-z^2}} \), where \( \sqrt{z} \) is with the branch cut of the interval \( (-\infty, 0] \), then on Bergman space \( A^2(\mathbb{D}) \),

\[
C^*_\varphi \left( \frac{1}{(2z-1)^2} \right) C \frac{1}{2-z^2} T^*_\varphi (2-z)^2 + C \frac{1}{2-z^2} T^* (2z) + C \frac{1}{2-z^2} T^* (2-z).
\]

3. **Compositions of n-th root functions and rational functions**

We can combine \( n \)-th root functions with some rational functions. For simplicity, we will first state the idea for the square root function.

**Proposition 2.** Let \( \psi(z) \) be a nonvanishing analytic self-map of the disk. Then \( \varphi(z) = \sqrt{\psi(z)} \) is an analytic self-map of the disk for some branch cut of the square root function, and on the Hardy space \( H^2(\mathbb{D}) \),

\[
C^*_\varphi f(\alpha) = C^*_\psi f(\alpha^2) + \alpha C^*_\psi g(\alpha^2),
\]

where \( g(z) = P \left[ \varphi(z) f(z) \right] \) and \( f(z) \in H^2(\mathbb{D}) \).

**Proof.** Note that by (2.4) with \( z \) replaced by \( \varphi(z) \),

\[
C^*_\varphi f(\alpha) = \langle f(z), K_\alpha(\varphi(z)) \rangle = \frac{1}{1 - \overline{\varphi} \varphi(z)} + \langle f(z), \frac{\overline{\alpha \varphi(z)}}{1 - \overline{\varphi} \varphi(z)} \rangle = C^*_\psi f(\alpha^2) + \alpha \left[ C^*_\psi T^*_\psi f \right] (\alpha^2).
\]

The proof is complete. 

The formulas for evaluating \( C^*_\varphi f(\alpha) \) for a rational symbol \( \psi \) were first derived in [15]. They are valid on some open subset of \( \mathbb{D} \); see also simple proofs and some refined interpretations of the formulas from [5].

We briefly recall some basic concepts following [5]. Let \( \hat{\mathbb{C}} \) denote the extended complex plane (i.e., the Riemann sphere). For a rational function \( \psi(z) \) with degree \( d \), associated with \( \psi(z) \) is its “exterior” map \( \psi_e(z) = \rho \circ \psi \circ \rho \), where \( \rho(z) = 1/z \) is the inversion of the unit circle. For each \( w \in \hat{\mathbb{C}} \), the inverse image \( \psi_e^{-1}(\{w\}) \) has, counting multiplicities, exactly \( d \) distinct points. If \( \psi_e^{-1}(\{w\}) \) has \( d \) distinct points, we say \( w \) is a regular value of \( \psi_e \). Since \( \psi_e \) is a rational function, all but finitely many points of \( \hat{\mathbb{C}} \) are regular values of \( \psi_e \). Let \( \text{reg}(\psi_e) \) denote the set of regular values of \( \psi_e \). Let \( \{\sigma_j(z)\}_{j=1}^d \) be \( d \) distinct branches of \( \psi_e \) which are defined on some neighborhood of any regular point of \( \psi_e \) in \( \mathbb{D} \). Then the formula for \( C^*_\varphi f(\alpha) \) is as follows [15, 5].

**Theorem 3.1.** Let \( \psi(z) \) be a rational function of degree \( d \) that is also a self-map of \( \mathbb{D} \). Suppose \( z_0 \in \mathbb{D} \) is a regular value of \( \psi_e \) and \( V \subseteq \mathbb{D} \) is any connected neighborhood
Theorem 3.2. Consider a function $\psi(z)$, as defined in the statement of Theorem 3.1. In addition, assume $\psi(z)$ is nonvanishing on $\mathbb{D}$. Let $\varphi(z) = \sqrt[n]{\psi(z)}$ be an analytic self-map of the disk for some branch cut of the square root function. Then, for any $f \in H^2(\mathbb{D})$,

(a) If $\psi(\infty) \neq \infty$, then for all $z^2 \in V \setminus \{1/\psi(\infty)\},$

$$C^*_{\varphi} f(z) = \frac{f(0)}{1 - \psi(\infty)z^2} + z^2 \sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z)} f(\sigma_j(z^2))$$

(b) If $\psi(\infty) = \infty$, then for all $z \in V, f \in H^2(\mathbb{D})$

$$C^*_{\psi} f(z) = \begin{cases} 
\sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z)} f(\sigma_j(z^2)) & \text{if } z \neq 0 \\
 f(0) & \text{if } z = 0
\end{cases}$$

Combining Proposition 2 with the above theorem, we have the following result.

Theorem 3.2. Consider a function $\psi(z)$, as defined in the statement of Theorem 3.1. In addition, assume $\psi(z)$ is nonvanishing on $\mathbb{D}$. Let $\varphi(z) = \sqrt[n]{\psi(z)}$ be an analytic self-map of the disk for some branch cut of the square root function. Then, for any $f \in H^2(\mathbb{D})$,

(a) If $\psi(\infty) \neq \infty$, then for all $z^2 \in V \setminus \{1/\psi(\infty)\},$

$$C^*_{\varphi} f(z) = \frac{f(0)}{1 - \psi(\infty)z^2} + z^2 \sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z)} f(\sigma_j(z^2)) + \frac{zg(0)}{1 - \psi(\infty)z^2} + z^3 \sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z^2)} g(\sigma_j(z^2))$$

where $g(z) = P \left[ \varphi(z) f(z) \right]$.

(b) If $\psi(\infty) = \infty$, then for all $z^2 \in V$,

$$C^*_{\psi} f(z) = \begin{cases} 
\sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z)} f(\sigma_j(z^2)) + z^3 \sum_{j=1}^{d} \frac{\sigma_j(z^2)}{\sigma_j(z^2)} g(\sigma_j(z^2)) & \text{if } z \neq 0 \\
 f(0) & \text{if } z = 0
\end{cases}$$

There is a more general way to compose functions. Namely, let $\varphi = \omega \left( \sqrt[n]{\psi(z)} \right)$ where $\omega$ and $\psi$ are two rational self-maps of $\mathbb{D}$. Furthermore, $\psi(z)$ is nonvanishing, so $\sqrt[n]{\psi(z)}$ is a well-defined self-map of $\mathbb{D}$. Then

$$C^*_{\varphi} = C^*_{\sqrt[n]{\psi(z)}} C^*_{\omega(z)}$$

Combining the formulas of $C^*_{\omega(z)}$ and $C^*_{\sqrt[n]{\psi(z)}}$ as in above two theorems will yield a formula for $C^*_{\varphi}$ (quite complicated, though). However, even when both $\omega$ and $\psi$ are linear fractional maps, as seen by the formula in Theorem 2.3, this approach leads to a formula for $C^*_{\varphi}$ containing terms such as

$$T_{\psi_1} C_{\varphi_1} T_{\omega_1} T_{\psi_2} C_{\varphi_2} T_{\omega_2}.$$

It is intriguing that, by working directly with the reproducing kernel, we get a simpler formula.

Theorem 3.3. Let $L(z) = \frac{az + b}{cz + d}$ and $L_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ be two linear fractional self-maps of $\mathbb{D}$ such that both $\sqrt[n]{L(z)}$ and $\varphi(z) = L_1(\sqrt[n]{L(z)})$ are self-maps of the disk. Then, on the Hardy space $H^2(\mathbb{D})$,

$$C^*_{\varphi(z)} = \sum_{k=0}^{n-1} T_{(d_1 - b_1 z)(z - b_1(z))^{n-k}} C_{L_1(z)^n} T_{(c_1 z + d_1)(cz + d) L(z)^k/n}.$$
On Bergman space $A^2(\mathbb{D})$,

$$C^*_{\varphi(z)} = \sum_{k=0}^{n-1} T \frac{(k+1)T_1(z)z^k}{2n-k} \frac{C_{\varphi(z)}}{2n-k} + \sum_{k=n}^{2n-2} T \frac{(2n-k)z^k}{2n-k} \frac{C_{\varphi(z)}}{2n-k}.$$ 

Proof. As in the proof of Theorem 2.3, the formula follows from the following computation of the kernel function:

$$K_\alpha(z) = \frac{1}{\sqrt{\varphi(z)}} = \frac{1}{1 - \alpha L_1(\sqrt{L(z)})} = \frac{1}{d_1 - b_1 \alpha(1 - L_1(\sqrt{L(z)}) \frac{L_1(z)^k}{\sqrt{L(z)^k}} = \frac{1}{d_1 - b_1 \alpha}\sum_{k=0}^{n-1} \frac{L_1(z)^k}{1 - \alpha L_1(\sqrt{L(z)})^k}.$$ 

where the third equality is by (2.2), the fourth equality is by (2.4) with $\alpha$ replaced by $L_1(\alpha)$, and the fifth equality is by (2.2) again — using $L_1(\alpha)^n$ in place of $\alpha$. The proof on $A^2(\mathbb{D})$ is similar by combining the above computation and the computation in the proof of Proposition 1.

Example 3.4. Let

$$\varphi(z) = \frac{1}{2 - \sqrt{\frac{1-z}{2}}} = L_1(\sqrt{L(z)})$$

Then, on Hardy space $H^2(\mathbb{D})$,

$$C^*_{\varphi(z)} = T \frac{z-1}{2-\sqrt{\frac{1-z}{2}}} C \frac{1}{2-\sqrt{\frac{1-z}{2}}} + T \frac{1}{2-\sqrt{\frac{1-z}{2}}} C \frac{1}{2-\sqrt{\frac{1-z}{2}}}$$

On Bergman space $A^2(\mathbb{D})$,

$$C^*_{\varphi(z)} = T \frac{z-1}{2-\sqrt{\frac{1-z}{2}}} C \frac{1}{2-\sqrt{\frac{1-z}{2}}} + T \frac{1}{2-\sqrt{\frac{1-z}{2}}} C \frac{1}{2-\sqrt{\frac{1-z}{2}}}$$

4. Inverse functions of finite Blaschke products

Note that $\sqrt{z}$ cannot be made into an analytic self-map of the disk, but $\sqrt{L(z)}$ is an analytic self-map of the disk for any nonvanishing linear fractional map $L(z)$. The $n$-th root function $\sqrt[n]{z}$ can be thought as an inverse of $z^n$. Note that $z^n$ is an inner function. In this section we extend the earlier results to get a formula for the adjoint of a composition operator whose symbol is an inverse function of a finite Blaschke product. The key is a formula similar to (2.4) for a finite Blaschke product. Incidentally, we remark, a (not so explicit) formula for $C^*_{\varphi(z)}$ where $\varphi$ is a finite Blaschke product was obtained by McDonald [20] before the general case of rational symbols [15, 5].
Let $\beta \in \mathbb{D}$ and let $\varphi_\beta(z)$ be the automorphism of the disk,
\begin{equation}
\varphi_\beta(z) = \frac{\beta - z}{1 - \beta z}, \quad z \in \mathbb{D}.
\end{equation}
For a finite Blaschke product $\theta(z)$, we have the following formula. This formula is also useful in proving an operator identity involving $m$-isometries, see Lemma 2.2 in [13].

**Lemma 4.1.** Let $\theta(z) = \lambda \prod_{k=0}^{n-1} \varphi_{\beta_k}(z)$ be a finite Blaschke product with $|\lambda| = 1$. Then
\[
1 - \overline{\theta(\alpha)}\theta(z) = (1 - |\alpha|^2) \prod_{k=0}^{n-1} \overline{\theta_k(\alpha)} \theta_k(z)
\]
for some analytic (rational) functions $\theta_k(z)$.

**Proof.** By a direct calculation,
\[
1 - \overline{\varphi_{\beta(\alpha)}(\alpha)}\varphi_{\beta}\varphi(z) = \left(1 - |\beta|^2\right) \left(1 - \overline{\alpha z}\right)^{-1} \left(1 - \overline{\beta z}\right)^{-1}.
\]
Now for a finite Blaschke product $\theta(z) = \lambda \prod_{k=0}^{n-1} \varphi_{\beta_k}(z)$, let
\begin{equation}
\theta_0(z) = \left(1 - |\beta_0|^2\right)^{1/2} \left(1 - \overline{\beta_0 z}\right)^{-1},
\end{equation}
\begin{equation}
\theta_k(z) = \left(1 - |\beta_k|^2\right)^{1/2} \left(1 - \overline{\beta_k z}\right)^{-1} \prod_{j=0}^{k-1} \varphi_{\beta_j}(z), \quad k = 1, \ldots, n - 1.
\end{equation}
Then
\[
1 - \overline{\theta(\alpha)}\theta(z) = 1 - \prod_{k=0}^{n-1} \overline{\varphi_{\beta_k}(\alpha)} \varphi_{\beta_k}(z)
\]
\[
= \prod_{k=0}^{n-1} \prod_{j=0}^{k-1} \left(1 - \overline{\varphi_{\beta_k}(\alpha)} \varphi_{\beta_k}(z)\right) \prod_{j=1}^{k-1} \varphi_{\beta_j}(z)
\]
\[
= (1 - \overline{\alpha z}) \sum_{k=0}^{n-1} \overline{\theta_k(\alpha)} \theta_k(z).
\]
This completes the proof. \qed

As shown in Proposition 12 in [5], the inverse branch of $\theta(z)$ is in general not defined on the whole unit disk $\mathbb{D}$. But as in the case $\theta(z) = z^n$, if $\sigma(z)$ is an inverse branch of $\theta(z)$ defined on an open subset of the disk, $\sigma(L(z))$ is a self-map of the disk if $L(z)$ maps the open unit disk into that open subset. Let $\sigma(z)$ be an inverse branch of $\theta(z)$ such that $\sigma(L(z))$ is a self-map of the disk. Then we have the following formula for the adjoint of $C_{\sigma(L(z))}$:

**Theorem 4.2.** Let $\theta(z)$ be a finite Blaschke product. Let $\sigma(L(z))$ be a self-map of the disk such that $\theta(\sigma(L(z))) = L(z)$. Then on the Hardy space $H^2(\mathbb{D})$,
\begin{equation}
C_{\sigma(L(z))}^* = \sum_{k=0}^{n-1} T_{\theta_k(z)}^* C_{\overline{\theta(\alpha)}} \overline{T_{\theta_k(z)}^*} \overline{T_{(cz+d)\theta_k(\sigma(L(z)))}}^\ast
\end{equation}
Proposition 3. Let \( \theta(z) \) be a finite Blaschke product. Let \( \sigma(L(z)) \) be a self-map of the disk such that \( \theta(\sigma(L(z))) = L(z) \). Then on the Bergman space \( A^2(\mathbb{D}) \),

\[
C_{\sigma(L(z))}^* f(\alpha) = \left\langle C_{\sigma(L(z))}^* f(z), K_\alpha(z) \right\rangle = \left\langle f(z), K_\alpha(\sigma(L(z))) \right\rangle
\]

Then

\[
C_{\sigma(L(z))}^* f(\alpha) = \sum_{k=0}^{n-1} \frac{\theta_k(\alpha)}{1 - \theta(\alpha)\theta(z)} = \frac{1}{1 - \alpha z} = K_\alpha(z).
\]

Then

\[
C_{\sigma(L(z))}^* f(\alpha) = \left\langle C_{\sigma(L(z))}^* f(z), K_\alpha(z) \right\rangle = \left\langle f(z), K_\alpha(\sigma(L(z))) \right\rangle
\]

\[
= \sum_{k=0}^{n-1} \frac{\theta_k(\alpha)}{1 - \theta(\alpha)\theta(z)} = \sum_{k=0}^{n-1} \frac{\theta_k(\alpha)}{1 - \theta(\alpha)\theta(z)}
\]

\[
= \sum_{k=0}^{n-1} \frac{\theta_k(\alpha)}{c - d \theta(\alpha)} = \frac{\theta_k(\alpha)}{c - d \theta(\alpha)}
\]

\[
g_k(z) = P \left[ (cz + d)\theta_k(\sigma(L(z)))f(z) \right], \quad \frac{1}{1 - L^*(\theta(\alpha))z}
\]

where the fifth equality is by (2.2), and

\[
g_k(z) = P \left[ (cz + d)\theta_k(\sigma(L(z)))f(z) \right] = T_{(cz+d)\theta_k(\sigma(L(z)))}^* f.
\]

The formula (4.3) follows from the above calculation. \( \square \)

We have a similar result on the Bergman space. The proof is by a combination of ideas from the proofs of Proposition 1 and Theorem 4.2.

Proposition 3. Let \( \theta(z) \) be a finite Blaschke product. Let \( \sigma(L(z)) \) be a self-map of the disk such that \( \theta(\sigma(L(z))) = L(z) \). Then on the Bergman space \( A^2(\mathbb{D}) \),

\[
(4.4) \quad C_{\sigma(L(z))}^* = \sum_{k=0}^{n-1} T_{\theta_k(z)} \cdot \frac{\theta_k(\alpha)}{c - d \theta(\alpha)}
\]

\[
+ 2 \sum_{k=0}^{n-1} \sum_{\ell=1}^{n-1} T_{\theta_k(z)\theta_{\ell}(z)} \cdot \frac{\theta_k(\alpha)}{c - d \theta(\alpha)}
\]

We next give an example to illustrate the above results.

Example 4.3. We chose the degree 2 Blaschke product to be the same one used in Example 6 of [5]. Let

\[
\theta(z) = \frac{1 - 2z}{2 - z}.
\]

Then the two branches of the inverses of \( \theta(z) \) are

\[
\sigma_1(z) = \frac{1 + z + \sqrt{\Delta(z)}}{4} \quad \text{and} \quad \sigma_2(z) = \frac{1 + z - \sqrt{\Delta(z)}}{4},
\]

where \( \Delta(z) = z^2 - 14z + 1 \). If we take the square root function in these formulas to have its branch cut along the positive real axis, then \( \sqrt{\Delta(z)} \) is not analytic on whole unit disk. However, for \( L(z) = (2 - z)/3, \sqrt{\Delta(L(z))} \) is analytic in the disk, since

\[
\Delta(L(z)) = \left( \frac{2 - z}{3} \right)^2 - 14 \left( \frac{2 - z}{3} \right) + 1 = \frac{1}{9}(z^2 + 38z - 71)
\]
has two roots, approximately, 1.78 and $-39.78$, so $\sqrt{\Delta(L(z))}$ is analytic on the complex plane cut by the rays $(-\infty, -39.78]$ and $[1.78, \infty)$. Note that
\[
\varphi = \sigma_1(L(z)) = \frac{5 - z + \sqrt{z^2 + 38z - 71}}{12}, \quad \theta_0(z) = 1, \quad \theta_1(z) = \frac{\sqrt{3z}}{2 - z}.
\]

Therefore on the Hardy space $H^2(\mathbb{D})$,
\[
C_{\varphi}^* = T_\frac{9}{3 - 2\theta(z)} C - \frac{\theta_1(z)}{3 - 2\theta(z)} + T_\frac{\theta_1(z)}{3 - 2\theta(z)} C - \frac{\theta_1(z)}{3 - 2\theta(z)} T_{3\theta_1(\sigma_1(L(z)))}^*.
\]

On the Bergman space $A^2(\mathbb{D})$,
\[
C_{\varphi}^* = T_{\frac{9}{3 - 2\theta(z)}} C - \frac{\theta_1(z)}{3 - 2\theta(z)} + T_{\frac{\theta_1(z)}{3 - 2\theta(z)}} C - \frac{\theta_1(z)}{3 - 2\theta(z)} T_{3\theta_1(\sigma_1(L(z)))}^* + 2T_{\frac{\theta_1(z)}{3 - 2\theta(z)}} C - \frac{\theta_1(z)}{3 - 2\theta(z)} T_{3\theta_1(\sigma_1(L(z)))}.
\]

REFERENCES


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