Subsidization to Induce Tipping

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Abstract

In binary choice games with strategic complementarities and multiple equilibria, we characterize the minimal cost subsidy program to guarantee agents play the Pareto optimal equilibrium. These subsidies are generally asymmetric, whether or not agents are identical and even if private values are anonymous.

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1 Introduction

This paper discusses subsidy-based policies to tip critical mass games from an inferior outcome to a more desirable outcome. Examples of these games from Thomas Schelling’s (1978) well known work on this topic include urban segregation, faculty seminar participation, and the use of helmets in professional hockey. Other examples include interdependent security (Heal and Kunreuther, 2003), poverty traps (Azariadas, 1996), urban development (Rauch, 1993), and some examples of public good provision. The possibility for tipping arises in games which exhibit supermodularity, as shown by Kunreuther and Heal (2010), who characterize the minimal tipping set needed to lead agents from one equilibrium to another.\(^1\)

The games analyzed here are binary strategy supermodular games. These games have two Pareto-ranked symmetric equilibria: one with all agents playing zero and the other, preferable equilibrium with all agents choosing one. For example, in the case of faculty seminar participation, the equilibrium with high attendance is preferable to the other equilibrium.

We characterize the optimal subsidy program that eliminates the inferior equilibrium in these games. We exclude by assumption the possibility of taxing the agents based on their actions. If taxing was a viable option, an infinite tax on the inferior action would solve the problem. However, in practice, an agency with a fixed budget but without the authority to set taxes may seek to induce tipping using subsidies in the minimal cost manner.

Two kinds of subsidy schemes are considered in Section 2: uniform subsidies and asymmetric subsidies. A uniform subsidy scheme involves the same payment for every agent who chooses the Pareto optimal strategy. Asymmetric subsidy schemes pay different amounts to different agents who choose the Pareto optimal strategy. The primary result of this paper is that asymmetric subsidy programs will eliminate the inferior equilibrium at less expense to the government agency than a uniform subsidy program, whether or not all agents are identical and even if private values are anonymous.

The asymmetric subsidy scheme which we describe below contrasts with the symmetric Pigouvian subsidy to correct inefficiencies in the presence of positive externalities. The distinction arises because, when one agent changes his or her strategy, it affects not just the total payoff to the other agents (as in the traditional Pigouvian situation) but also the marginal payoff that other agents receive from changing their own choice.

The results are applicable to any problem where a set of agents could get stuck at an inferior equilibrium leading to some sort of government intervention. In Section 3, we specifically apply our results to the interdependent security problem from Kunreuther and Heal (2003), although this example is not meant to exclude other possible applications.

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\(^1\)See Topkis (1998) for a comprehensive coverage of supermodular games.
2 The Model

Consider a simultaneous move game with a finite set of players $I = \{1, ..., I\}$. Denote by $i$ an arbitrary agent. Each agent has the binary strategy space $\{0, 1\}$. Denote player $i$’s strategy by $a_i \in \{0, 1\}$. Denote player $i$’s payoff by $u_i(a)$ for $a = (a_1, ..., a_I)$, or equivalently by $u_i(a_i, a_{-i})$ where $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_I)$.

The following assumption of increasing differences makes the game supermodular.

Assumption 1 $\forall i \in I$ and $\forall (a_{-i}, \tilde{a}_{-i}) \in \{0, 1\}^{I-1} \times \{0, 1\}^{I-1}$ such that $\tilde{a}_{-i} \geq a_{-i}$:

$$u_i(1, \tilde{a}_{-i}) - u_i(0, \tilde{a}_{-i}) \geq u_i(1, a_{-i}) - u_i(0, a_{-i}).$$

We formalize the fact that the game has two symmetric equilibria.

Assumption 2 The action $I$-tuples $a^0 = (0, ..., 0)$ and $a^1 = (1, ..., 1)$ are Nash equilibria of the game, where $a^1$ Pareto-dominates $a^0$.

We aim to characterize the minimal cost subsidization program which tips the game to $a^1$, making $a^1$ the unique Nash equilibrium. We assume that only positive subsidies can be used.

2.1 Identical Players

We start by considering games in which all players have identical payoff functions:

Assumption 3 (Identical Players) $\forall i \in I$, $u_i(a) = U(a_i, A_{-i})$ where $A_{-i} = \sum_{j \in I \setminus i} a_j$.

In Section 2.2, we relax this assumption and investigate optimal subsidies with heterogeneous players.

The minimal cost subsidy scheme which tips the game is asymmetric. An asymmetric subsidy scheme will work as follows. The government defines $I$ subsidies for choosing strategy 1. Denote the $t$-th subsidy by $S(t)$. An individual who chooses 1 knows they will receive a subsidy but does not know the value of the subsidy until all actions are observed. Suppose that $k$ agents choose 1. Then, the government will pay $k$ subsidies: $S(1), S(2), ..., S(k)$. The specific subsidy that an agent receives may be assigned arbitrarily.

\[\text{For vectors } x, y \in \mathbb{R}^n, \text{ the relation } x \geq y \text{ means } x_k \geq y_k \forall k \in \{1, 2, ..., n\}.\]
Since agents who choose 1 are unsure which subsidy they will receive, our notion of equilibrium requires that agents receive a subsidy sufficient to enforce the choice of 1 ex-post, after subsidies are assigned. This implies that the results are robust to any degree of risk aversion among agents, a feature which appears quite important in the context of policy implementation.\(^3\)

Assumptions 1 and 2 imply that the benefit to choosing 1 \((U(1, A_{-i}) - U(0, A_{-i}))\) is smallest (and negative) when all other agents choose 0 and increases as more agents choose 1 until at some point (i.e. the tipping point) it becomes positive. The idea behind the minimum cost subsidy is to set the first subsidy equal to the loss from choosing 1 when all other agents choose 0, the second subsidy equal to the loss from choosing 1 when exactly one other agent chooses 1, and so forth until the benefit from choosing 1 becomes positive, at which point all remaining subsidies are set equal to 0. This means that anyone who chooses 1 knows that, if the tipping point is not reached, the smallest possible subsidy they receive will exactly compensate them for the loss from choosing 1 instead of 0. If the tipping point is reached, the agent is happy to choose 1 even without a subsidy. Thus, choosing 1 becomes the dominant strategy for all agents.

Define
\[
\Lambda(k) = U(0, k) - U(1, k) \forall k \in \{0, ..., I - 1\}.
\]

The following proposition identifies the minimal cost subsidization program which tips the game to \(a^1\).

**Proposition 1** Under assumptions 1-3, the minimal cost subsidy scheme which eliminates all pure strategy equilibria except for \(a^1\) is \(S = (S(1), S(2), ..., S(I))\), where
\[
S(t) = \max\{\Lambda(t - 1), 0\} \quad \forall t \in \{1, 2, ..., I\}.
\]

Further, \(S(t) \geq S(t + 1) \quad \forall t \in \{1, 2, ..., I - 1\}\).

**Proof.** Based on Assumption 1, \(\Lambda(t - 1) \geq \Lambda(t) \quad \forall t \in \{1, 2, ..., I - 1\}\). This implies that \(\max\{\Lambda(t - 1), 0\} \geq \max\{\Lambda(t), 0\}\) which means that \(S(t) \geq S(t + 1) \quad \forall t \in \{1, 2, ..., I - 1\}\).

To show that \(S\) is sufficient to tip the game, suppose to the contrary that under the scheme \(S\) there is a Nash equilibrium, denoted \(a'\), other than \(a^1\). It is clear that \(a'\) cannot be such that at least \(\tau = \min\{t \in I | S(t) = 0\} - 1\) agents play 1 because it is optimal for all agents to choose 1 when at least \(\tau\) agents play 1. Instead, suppose \(a'\) is such that \(t < \tau\) agents play 1. Starting from \(a'\), the smallest possible subsidy an additional agent who chooses 1 receives under \(S\) is \(S(t + 1) = \Lambda(t)\). Note that \(\Lambda(t) = U(0, t) - U(1, t)\) which is exactly enough to induce this agent to choose 1.\(^4\) Thus, this agent prefers to play 1, eliminating \(a'\)

\(^3\)We could instead assume that subsidies are given to any agent uniformly and agents are risk neutral. In this case, scheme \(S\) still tips the game to \(a^1\), but there might be lower cost schemes possible.

\(^4\)Strictly for the purpose of closure, we will assume that if the payoff is equal for an agent they will pick 1 over 0.
as an equilibrium. \( S \) is therefore sufficient to eliminate all equilibria except \( a^1 \).

To show that \( S \) tips the game at minimal cost, we show that any subsidy scheme \( R \) for which \( R(t) < S(t) \) for some \( t \in \{1, ..., I\} \) does not tip the game to \( a^1 \). Denote by \( t_* \), the smallest \( t \) such that \( R(t) < S(t) \). For all \( t < t_* \),

\[
U(1, t) + R(t + 1) \geq U(1, t) + S(t + 1) = U(0, t),
\]

so \( t - 1 \) agents find it optimal to choose 1. When \( t - 1 \) agents are choosing 1, an additional agent does not find it optimal to choose 1 because they may receive subsidy \( R(t) \), in which case their payoff is

\[
U(1, t - 1) + R(t) < U(1, t - 1) + S(t) = U(0, t - 1).
\]

Thus, \( t - 1 \) agents choosing 1 is an equilibrium under \( R \), and \( R \) does not tip the game. ■

We now identify the minimal cost uniform subsidy in which every agent is paid the same subsidy. Any uniform subsidy less than \( S(1) \) will not eliminate the \( a^0 \) equilibrium since no agent will find it optimal to switch from 0 to 1 when all other players choose 0. Proposition 1 implies \( S(1) \) is the greatest of all subsidies in \( S \). A uniform subsidy of \( S(1) \) will therefore eliminate the \( a^0 \) equilibrium and any possible equilibrium other than \( a^1 \). Thus, the minimal cost uniform subsidy is \( S(1) \).

**Corollary 1** Under assumptions 1-3, the minimal cost uniform subsidy scheme is \( S^u = (S(1), ..., S(1)) \).

Since \( S(1) \) is the largest subsidy in \( S \), \( S^u \) is clearly inefficient at tipping the game compared to \( S \).

### 2.2 Heterogeneous Players

We now relax Assumption 3, allowing agents to have heterogeneous payoff functions. In the interest of parsimony, we impose that the strategies of all players \( j \in I \setminus i \) affect player \( i \)’s payoff symmetrically. This is formalized in the following assumption.

**Assumption 4** \( \forall i \in I, u_i(a) = U_i(a_i, A_{-i}), \) where \( A_{-i} = \sum_{j \in I \setminus i} a_j \).

We envision anonymous subsidies which work as follows. Consider an arbitrary subsidy scheme \( R = (R(1), ..., R(I)) \). Agents choose to either play 0 and not receive a subsidy or play 1 and receive a subsidy. If \( k \) agents choose 1, then subsidies \( R(1), ..., R(k) \) are arbitrarily given to those \( k \) agents.\(^5\)

\(^5\)It does not matter how the subsidies are given out to \( k \) agents, as long as it is the first \( k \) subsidies \( R(1), ..., R(k) \).
Define
\[ \Lambda_i(k) = U_i(0, k) - U_i(1, k) \quad \forall k \in \{0, ..., I - 1\}. \]
Further, define \( \Delta(0) = \min_{i \in I} \Lambda_i(0) \). For all \( k \in \{1, ..., I - 1\} \), define
\[
\Delta(k) = \min_{s_{k-1} \in S_{k-1}} \left\{ \min_{i \in I \setminus s_{k-1}} \Lambda_i(k) \right\}
\]
where \( S_{k-1} \) is the set of all subsets of size \( k \) of \( I_{k-1} \), and \( I_{k-1} = \{i \in I| \Lambda_i(k - 1) = \Delta(k - 1)\} \).
It will be useful in the proof of the proposition that follows to define
\[
S_{k-1}^* = \arg \min_{s_{k-1} \in S_{k-1}} \left\{ \min_{i \in I \setminus s_{k-1}} \Lambda_i(k) \right\}.
\]

Define the subsidy \( \hat{S}(t) = \max\{\max_{k \in \{t-1, ..., I-1\}} \Delta(k), 0\} \forall t \in \{1, 2, ..., I\} \) and denote \( \hat{S} = (\hat{S}(1), ..., \hat{S}(I)) \). Below we formalize the fact that \( \hat{S} \) is the anonymous minimal cost subsidy scheme. We also characterize the anonymous minimal cost uniform subsidy, denoted by \( \hat{S}^u \).

**Proposition 2** Under assumptions 1, 2, & 4:

1. The anonymous minimal cost subsidy scheme to tip the agents from the equilibrium \( a^0 \) to \( a^1 \) is \( \hat{S} \);
2. The anonymous minimal cost uniform subsidy to tip the agents from the equilibrium \( a^0 \) to \( a^1 \) is \( \hat{S}^u = (\hat{S}(1), ..., \hat{S}(1)) \).

**Proof.** To show that \( \hat{S} \) tips the game, suppose to the contrary that under the scheme \( \hat{S} \), there is a Nash equilibrium, denoted \( a' \), other than \( a^1 \). The equilibrium \( a' \) cannot be such that at least \( \tau = \min\{t \in I| \hat{S}(t) = 0\} - 1 \) agents play 1 because the game is then tipped to \( a^1 \). Denote the set of \( t \) agents that play 1 in \( a' \) by \( \tilde{I}_t \), where \( t < \tau \). The minimal subsidy payment for one more agent to play 1 is \( \hat{S}(t + 1) = \max_{k \in \{t, ..., \tau - 1\}} \Delta(k) \). By definition, \( \max_{k \in \{t, ..., \tau - 1\}} \Delta(k) \geq \Delta(t) \) and \( \Delta(t) \geq \Lambda_i(t) \) for at least one agent \( i \in I \setminus \tilde{I}_t \). Therefore, \( \exists i \in I \setminus \tilde{I}_t \) such that
\[
\hat{S}(t + 1) \geq \Lambda_i(t) \iff U_i(1, t) + \hat{S}(t + 1) \geq U_i(0, t).
\]
The above inequality implies that agent \( i \) prefers playing 1 in \( a' \), a contradiction.

The scheme \( \hat{S}^u \) pays each agent who plays 1 at least as much as \( \hat{S} \), so \( \hat{S}^u \) must also tip the game.

To show that \( \hat{S} \) is the minimal cost subsidy scheme that tips to \( a^1 \), suppose to the contrary that subsidy scheme \( R \) tips the game and \( \sum_{t \in I} R(t) < \sum_{t \in I} \hat{S}(t) \). Then, \( R(t) <
\( \hat{S}(t) \) for some \( t \in \{1, \ldots, I\} \). Denote \( t^* = \min\{t \in I | R(t) < \hat{S}(t)\} \). Further, denote \( t^+ = \min \{t \in \{t, \ldots, I\} | \min_{k \in \{1, \ldots, t\}} R(k) < \Delta(t-1)\} \). To establish that \( t^+ \) exists, we show that the set \( \{t \in \{t, \ldots, I\} | \min_{k \in \{1, \ldots, t\}} R(k) < \Delta(t-1)\} \) is non-empty. Based on the definition of \( \hat{S}(t) \), there exists a \( t \geq t^* \) such that \( \hat{S}(t) = \Delta(t-1) \). For such a \( t \), trivially, \( \min_{t \in \{1, \ldots, I\}} R(t) \leq R(t) \). We add the previously stated fact \( R(t) < \hat{S}(t) \) and we have

\[
\min_{t \in \{1, \ldots, I\}} R(t) \leq R(t) < \hat{S}(t) = \Delta(t-1),
\]

which insures that \( t \) is in the set, and as a result \( t^+ \) exists.

If \( t^+ = 1 \), then \( R(1) < \Delta(0) = \min_{i \in I} \Lambda_i(0) \). This implies that \( U_i(0, 0) > R(1) + U_i(1, 0) \) \( \forall i \in I \) which means all agents choosing 0 is an equilibrium, a contradiction to \( a^1 \) as the unique Nash equilibrium.

Now take the case that \( t^+ > 1 \). Consider an equilibrium where \( t^+ - 1 \) agents play 1 and the other \( I - t^+ + 1 \) agents play 0. Since \( \Delta(t^+ - 2) \leq R(t) \forall t \in \{1, \ldots, t^+ - 1\} \), all sets of \( t^+ - 1 \) agents in \( S^*_t \) will find it optimal to play 1 when the other \( I - t^+ + 1 \) agents play 0. By construction of \( \hat{S}(t) \) and \( S^*_t \), for each of the remaining \( I - t^+ + 1 \) agents, denoted by \( j \), \( \Lambda_j(t^+ - 1) \geq \Delta(t^+ - 1) > \min_{i \in \{1, \ldots, t^+\}} R(t) \). Therefore, it is possible that each of these \( I - t^+ + 1 \) agents will receive less than the amount to compensate the switch from 0 to 1 \textit{ex post}. Each of these agents finds it optimal to play 0, a contradiction to \( a^1 \) as the unique Nash equilibrium.

The extension to \( \hat{S}_u \) is straightforward. If a uniform subsidy \( R < \hat{S}(1) = \max_{i \in I} \hat{S}(i) \) is used, then \( a^0 \) is still a Nash equilibrium because no single agent will want to defect to 1 and take \( R \). ■

3 Policy Implications

We conclude with an application of our results to the interdependent security problem from Kunreuther and Heal (2003). Their leading example is the decision by airlines to screen checked bags for bombs when bags transferred from other airlines go unscreened. They characterize the minimum-sized coalition of agents that can tip the game from an inferior equilibrium to the preferred equilibrium with high levels of security. We now show how a government can form a tipping coalition by subsidizing investment in security.

Let \( p_{ij} \) denote the probability that a bomb is allowed on airline \( i \) and explodes on airline \( j \). Let \( L \) denote the loss from a plane exploding, let \( Y_i \) denote airline \( i \)'s income, and let \( c_i \) denote the cost of a security system which makes \( p_{ij} = 0 \ \forall j \) but has no effect on \( p_{ji} \).

We consider the simple case where there are \( N \) identical airlines (\( c_i = c, Y_i = Y, p_{ii} = p \ \forall i \) and externalities are symmetric (\( p_{ij} = q \ \forall i, j \)), although our results from Section 2.2
would apply in cases with heterogeneity. Let $K$ denote the number of other airlines choosing to invest. The payoff to investing in the security system is:

$$
\pi_I = \begin{cases} 
Y - c & \text{if } K = N - 1 \\
Y - c - qL \sum_{k=1}^{N-1-K} (1-q)^{k-1} & \text{if } K < N - 1
\end{cases}.
$$

The payoff to not investing is:

$$
\pi_N = \begin{cases} 
Y - pL & \text{if } K = N - 1 \\
Y - pL - (1-p)qL \sum_{k=1}^{N-1-K} (1-q)^{k-1} & \text{if } K < N - 1
\end{cases}.
$$

The benefit to investing is:

$$
\Delta(K) = \begin{cases} 
pL - c & \text{if } K = N - 1 \\
pL - c - pqL \sum_{k=1}^{N-1-K} (1-q)^{k-1} & \text{if } K < N - 1
\end{cases}.
$$

This is increasing in $K$ as a result of the last term. If $\Delta(0) < 0$ and $\Delta(N-1) > 0$, then the game has two symmetric Nash equilibria, everyone investing and everyone not investing. The optimal subsidy scheme to eliminate the non-investment equilibrium, following Proposition 1, is $S(i) = \max\{-\Delta(i-1), 0\}$ $\forall i \in (1, \ldots N)$.

This example represents just one case where asymmetric subsidies may provide the lowest cost way to tip a game to a preferred equilibrium. These results are also applicable to the provision of weaker link public goods (Cornes, 1993) where the benefit to contributing to a public good increases with the number of agents who contribute; to green markets with strategic complementarities across firms in the adoption of clean production technologies; and generally, to any binary choice game which exhibits supermodularity. The results justify the use of early-bird specials which reward the subset of agents who act quickly and whose actions subsequently induce other agents to follow.

References


