Demand Uncertainty Leads to Diverse Collusive Dynamics

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Abstract

We study collusive pricing in duopoly with uncertain demand and exogenous capacity constraints. In this setting, collusion using only symmetric pricing can limit collusive profits. We find that using asymmetric pricing in some demand states permits higher sustainable collusive profits over the entire range of demand states. Consequently, we show joint profit maximizing collusion can include both symmetric and two distinct types of asymmetric pricing on one equilibrium pricing path. Further, we derive conditions such that asymmetric pricing in a given state increases each firm’s individual discounted expected future profits.

Keywords: Demand Uncertainty; Collusion; Asymmetric Pricing; Capacity Constraints.


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1 Introduction

In many concentrated industrial settings firms pay substantial fixed costs to invest in capacity and then compete in prices over a long time horizon with fluctuating demand. For example, the hotel industry can often be viewed in this way as firms build hotels with a fixed number of rooms and demand fluctuates by day of week, by season, and due to other random factors like special events and macroeconomic fluctuations. Many industries with this basic structure, for example Aluminum, Copper, Steel, Sugar and Tin, have a documented history of collusive behavior (Levenstein and Suslow 2006). Of the 81 cartels found by the United States or European Union to have engaged in collusion from 1990 to 2007, roughly two thirds fall into some category of manufacturing, which commonly fit into the category of industries that require an up-front investment in capacity and compete in prices over time with demand fluctuation (Levenstein and Suslow 2011).

Industrial settings with some of these structural features have been widely studied in the theoretical literature. The seminal works on collusive pricing with non-stationary demand include: Green and Porter (1983), Rotemberg and Saloner (1986), Kandori (1991), Haltiwanger and Harrington (1991), and Bagwell and Staiger (1997). These papers study dynamic games without capacity limitations. A more recent literature integrates capacity constraints into dynamic games with non-stationary demand: Staiger and Wolak (1992), Fabra (2006) and Knittel and Lepore (2010).\(^1\) While the literature has produced important and interesting results, the models considered have restricted the firms to symmetric capacities and prices.\(^2\)

The goal of this paper is to better understand how collusive pricing might work in an industry with uncertain demand and asymmetric capacities that is free to collude using asymmetric prices. Asymmetric collusive pricing has been shown by Dechenaux and Kovenock (2011) to result in higher profits in a model with asymmetric capacities and stationary demand. The asymmetric pricing they find is such that the larger firm prices to maximize the residual profit, and the smaller firm charges the highest price that does not induce the larger firm to undercut.\(^3\) We investigate a richer environment with demand uncertainty to understand if firms can increase collusive profits by using asymmetric pricing in particular demand states. We show that the nature of collusive pricing can vary dramatically over the range of demand realizations. That is, a single joint profit maximizing collusive equilibrium path can involve periods of monopoly pricing, symmetric non-monopoly pricing, and two

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\(^1\)Both Fabra (2006) and Knittel and Lepore (2010) extend the deterministic demand cycle model of Haltiwanger and Harrington (1991); Fabra adds exogenous symmetric capacity constraints, while Knittel and Lepore add endogenous long-run capacities which can be asymmetric.

\(^2\)Knittel and Lepore (2010) is an exception in that symmetric pricing is imposed, but capacities are chosen endogenously. In their setting, the symmetry of the costs across firms combined with the symmetric pricing guarantees joint profit maximizing collusive capacities are symmetric.

\(^3\)Compte, Jenny and Rey (2002) introduced the model used in Dechenaux and Kovenock allowing asymmetric capacities, but restricting their analysis to subgame perfect equilibria with symmetric pricing.
distinct types of asymmetric pricing over the business cycle.

We consider a duopoly industry with uncertain future demand, exogenous capacities and a large range of residual demand rationing schemes.\(^4\) Allowing for a range of residual demand rationing schemes, from efficient rationing to proportional rationing, permits our results to apply to many industries without knowledge of the particularities of residual demand rationing. In the model, firms observe the current state of demand before choosing prices.\(^5\)

The character of collusive pricing is diverse in content. If the firms are sufficiently patient, then they will be able to sustain joint profit maximizing pricing in all demand states, which is symmetric. For sufficiently low discount factors, some demand states may have collusive pricing that is symmetric below the joint profit maximizing price or that is asymmetric.

We begin the analysis restricting collusive pricing to be symmetric to provide a baseline for comparison. With regard to symmetric collusion, we show that if symmetric collusive pricing is feasible in a given demand state, then there is a unique symmetric joint profit maximizing price. As the discount factor decreases, the first state in which symmetric collusive pricing drops below joint profit maximizing pricing, the critical state, is determined by the firms’ capacities. This critical state will be the largest demand state, following the “price war during booms” pattern of Rotemberg and Saloner (1986), if the firms have sufficiently large capacities. Otherwise, the critical demand state will be such that at least one firm’s capacity is less than or equal to the total demand at the monopoly price.

Our primary result is that there are two distinct types of asymmetric collusive pricing that can result in higher joint profit than symmetric collusion. The first type of asymmetric pricing (Type 1) is related to the collusive “judo pricing,” shown in Dechenaux and Kovenock (2011).\(^6\) In this case, the firm with the binding incentive compatibility constraint prices higher taking the residual demand, while the other firm prices sufficiently low to lower the binding firm’s gain from undercutting. We prove the following important property about this type of asymmetric pricing: if Type 1 asymmetric pricing results in higher joint profit than any symmetric pricing, then this asymmetric pricing must also give higher equilibrium expected profits to each firm individually. We provide precise conditions that guarantee this type of asymmetric pricing is part of joint profit maximizing collusion in a particular demand state.

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\(^4\) Analysis of a model with a range of residual demand rationing schemes is new to the collusive literature. The standard in the existing literature is to assume the “efficient” rationing scheme. We follow the work on two stage capacity pricing games of Lepore (2008, 2009, 2012) that provide synthesized results for models with multiple demand rationing schemes.

\(^5\) We follow the timing of demand uncertainty and price choices used by Rotemberg and Saloner (1986) and Staiger and Wolak (1992).

\(^6\) The concept of “judo pricing” was developed in Gelman and Salop (1983). They consider a model of entry where the entrant chooses capacity and price followed by an incumbent (with unlimited capacity) choosing price. The optimal strategy of the entrant is to choose capacity and price low enough that the incumbent is better off pricing as a monopolist and taking the residual demand than undercutting.
The second type of asymmetric pricing (Type 2) involves the firms pricing almost symmetrically with one firm undercutting by an arbitrarily small amount. This practice effectively rations the demand between the firms such that the firm who is having trouble colluding gets the primary demand up to its capacity in that state, eliminating that firm’s gain of defecting from collusion. Although we establish conditions such that Type 2 pricing will raise joint profit, this pricing can still result in lower profits than symmetric pricing for the firm with the slightly higher price. We also show that Type 2 collusive pricing can occur in a demand state in which joint profit maximizing collusion is sustainable, as pricing this way in non-binding state can serve to lower the incentive to defect from collusion in the binding state.

To further illustrate pricing patterns we calculate numerical examples with asymmetric capacities, both efficient and proportional demand rationing, and varying discount factors. With both rationing schemes we find asymmetric pricing is part of joint profit maximizing collusive pricing in particular demand states. There are two main insights that come from the examples: (i) pricing follows the “pricing wars during booms” pattern of Rotemberg and Saloner (1986) until capacity binds the larger firm’s gains from defection, and (ii) both symmetric pricing reductions below monopoly and asymmetric pricing are prevalent as the discount factors are reduced. Point (i) is that the “price war during boom” pattern is more prevalent in the pricing of the smaller capacity firm, which sustains collusion by reducing its price in the high demand states to lower the gain from defection of the larger firm. These high demand states will appear to be price wars in that the small firm is undercutting the large firm. Thus, the examples illustrate the generalization of the pattern to our model. Point (ii) is true for both demand rationing schemes used in our examples, but it is worth noting that proportional rationing leads to more prevalent asymmetric pricing than efficient rationing. This is simply based on the fact that the high price firm receives a larger demand with proportional rationing.

The remainder of the paper is organized as follows. In Section 2, we lay out the basic assumptions of our model. Section 3 deals with collusive pricing. We separately deal with the symmetric and asymmetric case and then provide a synthetic characterization. In Section 4, we present the numerical examples, and we conclude in Section 5.

2 Model Basics

Consider an industry with two firms that produce a single homogeneous product. The index \( i \) is used to identify an arbitrary firm, where \( i \in \{1, 2\} \). Throughout the paper we will use \( j \) to index the firm other than \( i \). Demand fluctuates over time according to random shocks. Let \( \omega_t \in [\underline{\omega}, \overline{\omega}] \) be the parameter for the demand state at time \( t \). We assume that the demand state is drawn independently from an identical distribution (\textit{iid}) each period, which is determined by the probability measure \( \mu \). The market demand function at time \( t \),
given the state $\omega_t$, is $D(\cdot, \omega_t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. The inverse demand for any time $t$, given the state $\omega_t$, is $P(\cdot, \omega_t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. The firms compete in prices over an infinite horizon such that the demand state in period $t$ is observed before firms choose their prices in that period.

The first assumption establishes the basic properties of the industry demand function.

**Assumption 1** For each $\omega$, there exists $\bar{P}(\omega) > 0$ such that $D(p_i, \omega) = 0$ for all $p_i \geq \bar{P}(\omega)$ and $D(p_i, \omega) > 0$ for all $p_i \in [0, \bar{P}(\omega))$. For any $\omega \in [\underline{\omega}, \overline{\omega}]$, $D(p_i, \omega)$ is twice-continuously differentiable, strictly decreasing and concave in $p_i$ on $(0, \bar{P}(\omega))$ and continuous and increasing in $\omega$. For any $p_i \in (0, \bar{P}(\omega))$, $D(p_i, \omega)$ is twice-continuously differentiable and strictly increasing in $\omega$ on $(\underline{\omega}, \overline{\omega})$.

Since Assumption 1 guarantees that if $\omega > \omega'$, then $D(p, \omega) > D(p, \omega')$ for all $p \in (0, \bar{P}(\omega))$, we will refer to the state $\omega$ as a larger demand state than $\omega'$.

We take each firm’s capacities to be exogenously fixed before the game begins. Firm $i$’s capacity is denoted by $x_i > 0$, which is an absolute limit on quantity firm $i$ can produce in any given period. Throughout the paper, without loss of generality, we index the two firms such that $x_1 \geq x_2$. Production up to capacity is cost-less. The two firms are assumed to share a common discount factor $\delta \in (0, 1)$.

Denote the unique monopoly price and quantity for any demand state $\omega$ by $\rho^m(\omega) = \arg \max p_i D(p_i, \omega)$ and $q^m(\omega) = D(\rho^m(\omega), \omega)$. The next assumption is made to guarantee that the monopoly price is increasing in the demand state.

**Assumption 2** For all $\omega \in (\underline{\omega}, \overline{\omega})$,

$$\rho^m(\omega) \frac{\partial^2 D(\rho^m(\omega), \omega)}{\partial p \partial \omega} + \frac{\partial D(\rho^m(\omega), \omega)}{\partial \omega} > 0.$$  

The following assumption is made to eliminate the trivial case of the model where capacities bind so tightly that there is no potential benefit from collusion

**Assumption 3** $x_1 + x_2 > \inf_{\omega \in [\underline{\omega}, \overline{\omega}]} q^m(\omega)$.

The analysis is focused on stationary equilibrium in the sense that there is a pricing strategy contingent on each demand state $\omega$. Hence, we define $p(\omega)$ as the pricing strategy based on the realization $\omega$. Denote by $p = (p(\omega))_{\omega \in [\underline{\omega}, \overline{\omega}]}$, the set of prices for all possible demand realizations. When referring to the pricing for a fixed state $\omega$, if it does not create confusion, then we will omit the argument and simply denote prices by $p = (p_1, p_2)$. 
2.1 Rationing and Payoffs

In pricing games with capacity constrained firms a specification must be made on how residual demand is rationed. To expand the applicability of our results, rationing of residual demand is allowed to follow any well behaved rule between proportional and efficient rationing. The demand of firm $i$ at prices $p$ in state $\omega$ formally defined below:

$$ Q_i(p, x, \omega) = \begin{cases} 
\min \{D(p_i, \omega), x_i\} & \text{if } p_i < p_j \\
\min \{\max \{D(p_i, \omega)/2, D(p_i, \omega) - x_j\}, x_i\} & \text{if } p_i = p_j \\
\max \{\min \{D^r_i(p, x_j, \omega), x_i\}, 0\} & \text{if } p_i > p_j 
\end{cases} $$

To simplify some of the conditions later in the text, in the case that firms’ prices are different we denote the sales for the lower price firm $q_i(p_i, x, \omega) = \min \{D(p_i, \omega), x_i\}$ and sales of the higher price firm $Q_i(p, x, \omega) = \max \{\min \{D^r_i(p, x_j, \omega), x_i\}, 0\}$.

Define two specific rationing schemes:

- **Efficient rationing:** “$r = e$”

  $$ D^e_i(p, x_j, \omega) = D(p_i, \omega) - x_j. $$

- **Proportional rationing:** “$r = p$”

  $$ D^p_i(p, x_j, \omega) = D(p_i, \omega) \left(1 - \frac{x_j}{D(p_j, \omega)}\right). $$

The following assumptions provide the basic properties of each firm’s residual demand.

**Assumption 4** $D^e_i(p, x_j, \omega) \in [D^e_i(p, x_j, \omega), D^p_i(p, x_j, \omega)]$. $D^e_i(p, x_j, \omega)$ is a continuous function and weakly decreasing in $p_i$ on $[0, \overline{P}(\omega)]$.

**Assumption 5** There is a unique residual demand maximizer

$$ f^e_i(p_j) = \arg \max \{p_iD^e_i(p_i, p_j, x_j, \omega) | p_i \geq p_j\}. $$

For all $p_j \geq \rho^m$, $f^e_i(p_j) = p_j$. For all $r$, the residual profit is strictly increasing in $p_i$ for $p_i \in [p_j, f^e_i(p_j)]$.

Denote by $\rho^r_j$, the smallest price such that $f^r_i(\rho^r_j) = \rho^r_j$. It will also be useful to denote the maximal residual profit by

$$ \pi^r_i(p_j, x_j, \omega) = \max \{p_iD^r_i(p_i, p_j, x_j, \omega) | p_i \geq p_j\}. $$
The next two assumptions are imposed only to control demand rationing enough to
ensure that in every demand state the joint profit maximizing pricing is symmetric. These
assumptions make asymmetric collusive pricing profitable only if it can ease the collusive
incentive compatibility constraints. Consequently, these assumptions make it more difficult
for asymmetric pricing to emerge. Note, both assumptions hold for the important cases of
efficient and proportional rationing.

**Assumption 6** For all \( \omega, x_j \) and \( p_i \in [p_j, P(\omega)] \),
\[
x_j + \frac{\partial \pi^*_i(p_j, x_j, \omega)}{\partial p_j} \leq 0.
\]

For the next assumption, define the price \( P^r_i \) such that \( D^r_i(P^r_i, p_j, x_j, \omega) = x_i \).

**Assumption 7** For all \( \omega, x_j \) and \( p_i \in [p_j, P(\omega)] \),
\[
\left( \frac{1}{x_i} \right) \frac{\partial D^r_i(P^r_i, p_j, x_j, \omega)}{\partial p_i} < \left( \frac{1}{x_j} \right) \frac{\partial D^r_i(P^r_i, p_j, x_j, \omega)}{\partial p_j}
\]

Denote by \( \pi_i(p, \omega) = p_iQ_i(p, x, \omega) \), the state profit of firm \( i \) and \( \pi(p, \omega) \) the sum of the
two firm’s profit. Similarly, denote the discounted expected payoff of firm \( i \) by \( V_i(p) = \frac{1}{1-\delta} \int \pi_i(p, \omega) d\mu \) and the sum of the firms discounted expected payoffs by \( V(p) \). We will
often omit the capacity argument in the notation of functions from this point forward to
avoid cumbersome notation.

### 3 Collusive pricing

We begin our analysis by establishing the collusive baseline of the joint profit maximizing
solution. The following proposition establishes existence and character of the joint profit
maximizing (from this point onward JM) solution. We impose the intuitive constraint that
neither firm makes zero profit with joint profit maximizing pricing.

**Proposition 1** For any fixed \( x \), the JM price in each state \( \omega \) is
\[
\rho^J(\omega) = \begin{cases} 
\rho^m(\omega) & \text{if } q^m(\omega) < x_1 + x_2, \\
\frac{x}{x_1 + x_2} P(x_1 + x_2, \omega) & \text{if } q^m(\omega) \geq x_1 + x_2.
\end{cases}
\]

(1)

The proof of this result is based on first showing that in each demand state there is a
unique symmetric JM price, and second using Assumptions 6 and 7 to rule out the possibility
of higher profit asymmetric pricing. The proofs of all results are located in the appendix.
It will be useful to denote the total JM quantity in state $\omega$ by $q'(\omega)$ and the JM profit for firm $i$ in state $\omega$ by $\pi_i'(\omega)$. We will also denote the symmetric JM price vector $p'_{ij}(\omega) = (p_{i}(\omega), p_j(\omega))$.

The set of collusive subgame perfect equilibria can be characterized using incentive compatibility constraints based on the gain of defecting from collusion and the future expected loss from punishment. The upper bound on the gain of defecting from pricing $p$ in state $\omega$ is

$$G_i(p, \omega) = \sup_{p'_i} \pi_i(p'_i, p_j, \omega) - \pi_i(p, \omega).$$

We assume that the punishment payoffs are fixed based on $x$ and $\omega$. This is a feature of punishment based on noncooperative reversion.\(^7\) We label the state $\omega$ punishment payoff as $\pi_i(\omega)$. The expected loss after a defection from collusive pricing $p$ can be written

$$L_i(p, \delta) = \frac{\delta}{1-\delta} \int [\pi_i(p, \omega) - \pi_i(\omega)] d\mu.$$

We study the stationary set of subgame perfect equilibrium that satisfy $G_i(\bar{p}, \omega) \leq L_i(\bar{p}, \delta)$ for both firms $i$ and all $\omega$. Formally define Joint Profit Maximizing Collusive (JC) pricing as

$$p^* \in \sup \{ V(p) | G_i(p, \omega) \leq L_i(p, \delta) \text{ for all } i \text{ and all } \omega \}.$$

We begin the analysis of collusion by establishing a folk theorem for this game.

**Proposition 2** For any fixed $x$, there exists a $\delta$ such that for all $\delta \in [\delta, 1)$ JM pricing is a subgame perfect equilibrium.

Now we turn attention to more detailed analysis of collusive pricing. To help keep the concepts clear, results for symmetric and asymmetric collusion are presented separately. First we provide some results specific to symmetric collusive pricing.

### 3.1 Symmetric pricing

In this section, consider collusion with symmetric pricing in all demand states and let $\rho$ denote a set of symmetric prices for all demand states. Denote the gain from defection and future expected loss from punishment in terms of symmetric pricing $G_i(\rho, \omega) = G_i(\rho, \rho, \omega)$ and $L_i(\rho, \delta) = L_i(\rho, \rho, \delta)$, respectively.

\(^7\)Lambson (1987, 1994) has shown that noncooperative reversion is the optimal penal code with efficient rationing. On the other hand, noncooperative reversion is not necessarily an optimal penal code with proportional rationing.
The primary result pertaining to symmetric collusive pricing is that if symmetric collusion is feasible, then unique symmetric joint profit maximizing collusive pricing in each state $\omega$ exists. We formally define the **Symmetric Joint Profit Maximizing Collusive (SC)** pricing as the symmetric pricing $\rho^*$ that maximizes joint profit out of the set of symmetric subgame perfect equilibria. Define $\tilde{\delta}$ as the lowest discount factor such that collusive symmetric pricing is feasible:

$$
\tilde{\delta} = \min \left\{ [0, \tilde{\delta}] \left| \begin{array}{c}
G_i(\rho, \omega) \leq L_i(\rho, \delta) \text{ and } \\
f_i(\rho, \omega) - \pi_i(\omega) \geq 0,
\end{array} \right. \text{ for some } \rho \text{ and all } i \right\}.
$$  \hfill (2)

**Proposition 3** There exists $\tilde{\delta} \in [0, \tilde{\delta}]$ such that for all $\delta \in [\tilde{\delta}, 1)$, there exists unique SC pricing.

In any state the SC symmetric price is simply the largest price that satisfies the incentive constraints for that state and is less than or equal to the JM price. SC pricing has this structure because increasing the symmetric price in any state increases that state’s joint profit and increases the future expected loss from punishment in all states.

Formally, define a critical state $\bar{\omega}$ such that for all $\delta$ close enough to $\tilde{\delta}$, $\rho^*(\omega) = \rho^I(\omega)$ for all $\omega \in [\omega, \bar{\omega}] \setminus \bar{\omega}$. We assume throughout analysis that there exists a single critical state. That is, for $\delta$ sufficiently close to $\tilde{\delta}$ the SC price will drop below JM pricing in exactly one state and remain at JM pricing in the other states.

**Proposition 4** Suppose that firm $i$’s constraint binds at $\tilde{\delta}$. The critical state $\bar{\omega}$ must be as follows:

(i) \quad $\bar{\omega} = \bar{\omega}$, if $x_i \geq q^I(\bar{\omega})$, \\
(ii) \quad $\bar{\omega}$ is such that $x_i \leq q^I(\bar{\omega})$, if $x_i < q^I(\bar{\omega})$.

Note that if the capacity of the firm with the binding constraint is larger than the total JM quantity, then the critical state is the largest demand state. For any other case, the critical state will not necessarily be the largest demand state. In contrast to symmetric pricing, we show in the next section that with asymmetric pricing, we may observe price reductions from JM pricing in a single state that is not $\bar{\omega}$.

### 3.2 Asymmetric pricing

At this point we introduce the possibility of asymmetric collusive pricing in one or more states. There are two distinct types of asymmetric pricing: (Type 1) $p_i > p_j$, where firm $i$ has a binding constraint in state $\omega$ at SC pricing, and (Type 2) $p_i > p_j$, where firm $i$ does not have a binding constraint in state $\omega$ at SC pricing. Both types of asymmetric pricing, as well as symmetric pricing, can appear on a single JC equilibrium path.
3.2.1 Type 1

Type 1 pricing is such that the larger firm prices higher and takes the residual demand. This pricing is related to, but distinct from, the asymmetric pricing in Dechenaux and Kovenock (2011). In our context the larger firm charges the higher price, but does not always charge the price that maximizes residual profit. The following proposition shows the conditions such that a state’s pricing will be this way to maximize joint profit.

**Proposition 5** Assume $\delta \geq \hat{\delta}$. Further suppose that $\delta$ is such that $\rho^s < \min\{p_2^s, \rho^l\}$, $x_2 \leq D(\rho^s, \omega)/2$, and $x_1 + x_2 > D(\rho^s, \omega)$, then there is asymmetric pricing $p_1 > p_2$ in the state $\omega$ that results in higher joint profit than the best symmetric collusive pricing. Further, there are always asymmetric collusive prices that weakly increase each firm’s discounted expected collusive profits relative to $\rho^s$.

Proposition 5 is a strong result about Type 1 collusive pricing. This proposition tells us that Type 1 asymmetric pricing is JC in any state $\omega$ and discount factor $\delta$ such that the smaller firm cannot serve half the SC market demand and the sum of the two firms’ capacities still covers the entire SC market. The most powerful part of this result is that these conditions also guarantee that each firm individually is better off with Type 1 pricing than any symmetric collusive pricing in state $\omega$. The profit increase for each firm over symmetric collusion is important to justify the feasibility of firms utilizing this type of collusion.

3.2.2 Type 2

We now consider a different kind of asymmetric pricing where one firm undercuts the other firm by an arbitrarily small amount. First consider a price reduction in the critical state $\hat{\omega}$. The next proposition establishes when asymmetric collusive pricing is JC pricing in the state $\hat{\omega}$ for $\delta < \hat{\delta}$, but sufficiently close to $\hat{\delta}$. Formally we will use the discount factor $\hat{\delta} - \epsilon$ for $\epsilon > 0$. We will use the notation $\hat{p}$ to denote prices in state $\hat{\omega}$.

**Proposition 6** Suppose that firm $i$ has the only binding incentive constraint at pricing $\hat{\rho}^l$ and discount factor $\delta$ in state $\hat{\omega}$. If

$$\hat{\rho}^l \left[ q_j(\hat{\rho}^l, \hat{\omega}) - Q_j^l(\hat{\rho}^l, \hat{\omega}) \right] < L_j(\rho^l, \hat{\delta}) + \mu(\hat{\omega}) \frac{\hat{\delta}}{1 - \hat{\delta}} \hat{\rho}^l \left[ Q_j^l(\hat{\rho}^l, \hat{\omega}) - Q_j(\hat{\rho}^l, \hat{\omega}) \right],$$

then at the discount factor $\hat{\delta} - \epsilon$ for small enough $\epsilon > 0$, there is asymmetric collusive pricing $\hat{\rho}_j = \hat{\rho}^l$ and $\hat{\rho}_i < \hat{\rho}^l$ that permits higher joint collusive expected profit than any symmetric pricing in state $\hat{\omega}$.
Type 2 asymmetric pricing, as described in the above proposition, involves firm \( i \), with the binding constraint, undercutting the JM price by an arbitrarily small amount, eliminating any gain from defection for firm \( i \). Since this type of asymmetric pricing achieves joint profits arbitrarily close to JM profits, it is always the most profitable pricing for a discount factor close enough to \( \hat{\delta} \).

Note that we have intentionally worked around the closure problem here; that there is no true maximal asymmetric pricing of this form. The closure problem is based on the fact that at any Type 2 asymmetric collusive prices, as firm \( i \) prices closer and closer to firm \( j \) the joint profit increases. To rectify this problem intuitively one can imagine the lower price firm pricing up to some arbitrarily small unit difference.

Type 2 pricing can instead occur in states other than \( \hat{\omega} \) to relieve the binding constraint in \( \hat{\omega} \). Particularly, if inequality (3) in Proposition 6 does not hold for state \( \hat{\omega} \), then it is possible that for some other state \( \hat{\omega} \) a similar undercut pricing can be used to loosen the constraint in the binding state \( \hat{\omega} \).

**Proposition 7** Suppose that firm \( i \) has the only binding incentive constraint at pricing \( \bar{\rho} \) and discount factor \( \hat{\delta} \) in state \( \hat{\omega} \). If \( \mu(\hat{\omega}) > 0 \) and

\[
\bar{\rho} \max \left\{ q_j(\bar{\rho}, \hat{\omega}) - Q_j(\bar{\rho}, \hat{\omega}), \max_{\omega \in [\hat{\omega} \rightarrow \hat{\omega}]} \left\{ q_j(\rho', \omega) - Q_j(\rho', \omega) \right\} \right\} < L_j(\bar{\rho}, \hat{\delta}) + \mu(\hat{\omega}) \left\{ \frac{3}{1 - \epsilon} \bar{\rho}' \left[ Q_j(\bar{\rho}, \hat{\omega}) - Q_j(\bar{\rho}, \hat{\omega}) \right] \right\},
\]

(4)

then at discount factor \( \hat{\delta} = \epsilon \) for \( \epsilon > 0 \) small enough, there is asymmetric collusive pricing \( \bar{\rho}_j = \bar{\rho}' \) and \( \bar{\rho}_i < \bar{\rho}' \) that permits higher joint collusive expected profit than any symmetric pricing in state \( \hat{\omega} \).

This is an interesting type of asymmetric pricing, as an arbitrarily small price undercut in a non-critical state is sufficient to keep JM profits possible in all other demand states, keeping the discounted joint collusive profit arbitrarily close to the JM discounted total profit.

The results pertaining to the two types of asymmetric collusion are substantively different. Propositions 6 and 7 on Type 2 pricing only address a single price movement away from JM pricing. Further, in contrast to the result on Type 1 pricing, if \( \delta \) is close enough to \( \hat{\delta} \), then Type 2 pricing necessarily will lower the profit of the higher price firm (\( j \)) below that of symmetric collusion. This is because the SC pricing is close to JM pricing and the demand of firm \( j \) is smaller with asymmetric pricing. More formally, the profit of firm \( j \) at \( \hat{\delta} - \epsilon \) for very small \( \epsilon > 0 \) with Type 2 asymmetric pricing is arbitrary close to \( \rho'Q_j(\rho', \omega) \) while the profit with SC pricing is approximately \( \rho'q_j(\rho', \omega) \). In order for Type 2 asymmetric pricing to loosen the binding constraint of firm \( i \), it must be that \( q_j(\rho', \omega) > Q_j(\rho', \omega) \). Thus, in spite of the fact that Type 2 pricing leads to higher joint profit than symmetric pricing, it can increase the total profit at the expense of lowering the profit one of the firm.
4 Numerical Examples

To provide a concrete illustration of the diverse nature of collusive pricing we parameterize the model and calculate numerical examples of the JC equilibrium that involve Type 1 asymmetric as well as symmetric pricing.

The numerical examples provide a more detailed picture of the collusive pricing patterns than can be shown analytically. We use the demand function $D(p, \omega) = \omega - 400p$ and calculate JC pricing in four demand states $\omega$, given as $\Omega = \{200, 300, 400, 500\}$. This demand function is the same specification to used in Haltiwanger and Harrington (1991) and Knittel and Lepore (2010). Each state is assumed to occur with equal probability. We impose the constraint that if possible, both firms make strictly positive profits in each state. Exogenous capacities are fixed at $x_1 = 500$ and $x_2 = 200$. These capacities were selected to provide an interesting mixture of symmetric and asymmetric Type 1 collusive pricing. The data is presented for discount factors in the range that JC pricing is neither one shot noncooperative pricing nor JM pricing. Punishment payoffs are calculated based on reversion to non-cooperative equilibrium from the point of a defection onward. We present JC pricing for the two prominent residual demand rationing rules, efficient and proportional rationing, in the following sections.

4.1 Efficient Rationing

Figure 1 shows the JC pricing of firms 1 and 2 with efficient rationing. At the discount factor 0.8, JM pricing is sustainable. The drop to 0.79 induces a symmetric price drop to 0.387 in the highest demand state 500. At the discounts 0.78 and 0.77, we get asymmetric pricing of Type 1 with firm 2 pricing at 0.284 and 0.274, respectively, and Firm 1 pricing at the residual profit maximizing price 0.375. For the discounts 0.76 and 0.75 we see the same Type 1 asymmetric pricing in state 500 with only firm 2’s price reducing further, but the firms are also forced to reduce their pricing in state 400. This price reduction is symmetric and at 0.75 we observe that pricing in state 400 is lower than in the 300 and 200 states.

Notice that the pricing pattern of the demand cycle gradually becomes somewhat countercyclical as the discount factor decreases. This is particularly true of Firm 2’s pricing and is driven by the gain from defection of Firm 1 (the larger firm). Firm 1’s relatively larger gain from defection in the high demand states drives Firm 2 to price asymmetrically lower in these states and results in a counter cyclical pricing pattern.

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8We find Type 2 pricing less empirically relevant since prices of the two firms are so close that it could appear as almost symmetric pricing. For this reason, we constructed numerical examples that exhibited the more economically interesting case of Type 1 asymmetric pricing.
Figure 1: Pricing with Efficient Rationing
4.2 Proportional Rationing

Proportional rationing gives the higher priced firm a better residual demand than efficient rationing. This explains the relative prominence of asymmetric pricing in the numerical example with this rationing scheme. Figure 2 plots JC pricing of firms 1 and 2. Overall, there is very little change in the pricing of firm 1 over the various discount factors. Since the same asymmetric pricing will lead to higher joint profits with proportional rationing than efficient, all but one of the observed price reductions is asymmetric. Also, since the residual profit maximizing price is the monopoly (JM) price, Firm 1 sticks at monopoly prices in all states until the two lowest discount factors. At the discount factors 0.82, 0.80, and 0.78 the only departure from JC pricing is Firm 2’s price in state 500, which drops down to 0.36, 0.34, and 0.32, respectively. At the discount factor of 0.76, firm 2 prices even lower in state 500 and firm 1 drop its price in state 500 to 0.54. We also observe a symmetric price drop in the pricing of state 400 to 0.35. An interesting thing happens between 0.76 and 0.74: the pricing in state 400 switches to asymmetric pricing as firm 2 drops its state 400 price to 0.215 and firm 1 increases its state 400 price back to the monopoly price. In state 500, both firm 1 and firm 2 price higher with discount factor 0.74 compared to 0.76. This transition, from lowered symmetric pricing to asymmetric pricing, is loosely predicted by Proposition 5, which specifies conditions such that if the discount factor decreases enough, driving feasible symmetric pricing is sufficiently low, then asymmetric pricing becomes more profitable.

5 Conclusion

We have established that in an industry with demand uncertainty and long run capacity investment that firms are able to increase the profitability of collusion by using asymmetric pricing in some demand states. Asymmetric pricing can increase profits in states such that only one firm has a binding incentive constraint at symmetric collusive pricing. In this case, there are two mechanisms by which the incentive constraint can be loosened.

The first type of asymmetric pricing is such that the non-binding firm prices lower than the binding firm. In this case, lowering the non-binding firm’s price decreases the gain from defection for the binding firm. As is formally stated in Proposition 5, the binding firm must be the larger capacity firm for this type of asymmetric pricing. Further, when this pricing increases joint profit it also increases the individual profit of each firm. This type of pricing appears very similar to a large firm acting as the price leader and the small firm acting as the discount supplier.

The second type of asymmetric pricing is such that the binding firm’s price is slightly lower than the non-binding firm’s price. This eliminates any gain from defection for the formerly binding firm by increasing the profit in that period to the defection profit. This type of asymmetric pricing increases the gain from defection of the other firm and, consequently, is
Figure 2: Pricing with Proportional Rationing
feasible only if the original “non-binding” firm has significant slack in its collusive incentive compatibility constraint.

6 Appendix

Proof of Proposition 1. The proof is done in two parts. First, we show that among symmetric prices there is a unique joint profit maximizing solution. Second, we show that no asymmetric pricing gives higher joint revenue than \( \rho^J \).

**Part 1:** Symmetric Joint Profit Maximizing Pricing

We can write the symmetric price joint profit maximization problem for any \( \omega \) as

\[
\arg \max_{\rho \in \{0, P\}} \{ \rho D(\rho, \omega) | D(\rho, \omega) \leq x_1 + x_2 \}.
\]

This is a standard constrained monopoly problem with strictly concave profit, where \( q^m(\omega) \leq x_1 + x_2 \) implies the solution is \( \rho^m(\omega) \). While if the demand constraint is binding \( q^m(\omega) \geq x_1 + x_2 \) the joint profit is maximized at the highest price such that \( D(\rho, \omega) \geq x_1 + x_2 \), which is exactly \( P(x_1 + x_2, \omega) \).

**Part 2:** We will show that no asymmetric pricing has higher joint profit than \( \rho^J \). Consider asymmetric prices \( p^J_j > p^J_i \). Suppose to the contrary that these asymmetric prices maximize joint profit. We will show a contradiction for each of the three possible cases.

**Case 1:** Joint profit is \( p^J_i D(\rho^J_i) \). In this case, it must be that \( p^J_i = \rho^m \), and firm \( j \) pricing also at \( \rho^m \) gives the same joint profit.

**Case 2:** Joint profit is \( p^J_i x_i + p^J_j D_j(\rho^J, x_i) \). To be maximal \( p^J_j = p^J_j(\rho^m) \). Note that if \( p^J_j > \rho^m \), then \( p^J_j(\rho^J) = \rho^m \) and prices are not asymmetric. Thus, the only possibility is that joint profit is \( p^J_i x_i + \pi^J_i(\rho^m, x_i) \) and \( p^J_i < \rho^m \). But based on Assumption 6, this function is strictly increasing in \( p^J_i \) on the interval \([0, \rho^m] \), a contradiction.

**Case 3:** Joint profit is \( p^J_i x_i + p^J_j x_j \). For these prices to be maximize joint profit it must be that \( p^J_j = P^J_j \). To be asymmetric must be such that \( p^J_i < \rho^J \). Thus, joint profit is \( p^J_i x_i + P^J_j x_j \) and \( p^J_i < \rho^J \). We show a contradiction by demonstrating that this function is strictly increasing in \( p^J_i \) on the interval \([0, \rho^m] \). Taking the derivative of this function with regards to \( p_i \) we have the expression

\[
x_i + \left( -\frac{\partial D_j(\rho^J, x_i, \omega)}{\partial p_i} \right) x_j > 0.
\]

The positive inequality is based on Assumption 7. ■
Proof of Proposition 2. Consider \( x_1 + x_2 > q^\omega(\omega) \) (The other case is not interesting since trivially \( \hat{\delta} = 0 \)). Define the discount

\[
\hat{\delta} = \max \{ \delta \in [0,1] \mid G_i(p^i', \omega) \leq L_i(p^i', \delta) \text{ for all } \omega \}.
\]

Since \( \pi_i^\delta(\omega) > \pi_i(\omega) \) for a positive measure of states and \( \pi_i^\delta(\omega) \geq \pi_i(\omega) \) for all \( \omega \), then \( \int (\pi_i^\delta(\omega) - \pi_i(\omega)) \, d\mu > 0 \). Thus we can write \( L_i(p^i', \delta) = [\delta/(1-\delta)] \int (\pi_i^\delta(\omega) - \pi_i(\omega)) \, d\mu \).

Notice this expression is continuous and strictly increasing in \( \delta \), and \( \lim_{\delta \uparrow 1} L_i(p^i', \delta) \to +\infty \).

\( G_i(p^i', \omega) \) is finite for all \( \omega \). This implies that \( \hat{\delta} \) is well defined and \( \hat{\delta} < 1 \).

Proof of Proposition 3. We begin by establishing the existence of \( \hat{\delta} \). The set in Equation 2 is a subset of a compact interval \([0, \hat{\delta}]\) and closed since both inequalities are weak and defined by continuous functions in \( \delta \) and \( \rho \). Thus, the set is compact and a minimum must exist.

Since \( L_i(\rho, \delta) \) is strictly increasing in \( \delta \), at any \( \delta' > \hat{\delta} \) all prices that satisfy the constraints at \( \hat{\delta} \) must also satisfy the constraints at \( \delta' \). Consequently SC pricing exists for all discounts in the interval \([\hat{\delta}, 1]\).

Next we establish uniqueness. From Proportion 2, for any \( \delta \in [\hat{\delta}, 1] \), \( \rho^* = \rho^i' \). We only need to consider the case that some constraints bind. At any symmetric price \( \rho \) the joint profit in any state \( \omega \) is \( \rho \min \{ D(\rho, \omega), x_1 + x_2 \} \). Since \( L_i(\rho, \delta) \) is weakly increasing in each state price \( \rho \), the problem for each state can be written independently as

\[
\max_{\rho \in [0, \rho^i']} \{ \rho \min \{ D(\rho, \omega), x_1 + x_2 \} \mid G_i(\rho, \omega) \leq L_i(\rho, \delta) \text{ for } i = 1, 2 \}.
\]

Each firm \( i \)'s state \( \omega \) profit function \( \pi_i((\rho, \rho), \omega) = \rho \min \{ \max \{ D(\rho, \omega)/2, D(\rho, \omega) - x_j \} \, x_i \} \) is continuous in \( \rho \). Notice that \( \sup_{p_i} \pi_i(p_i, \rho, \omega) = \max \{ \rho \min \{ D(\rho, \omega), x_1 \}, \pi_i((p_i, \rho), \omega) \} \).

Both functions in the maximum are continuous in \( \rho \), then the maximum of these two functions is continuous. Thus, both \( G_i(\rho, \omega) \) and \( L_i(\rho, \delta) \) are continuous in \( \rho \). Since the constraint set is composed of weak inequalities of continuous functions, the set is closed. The fact that \( \rho \) is bounded in \([0, \rho^i']\) makes the constraint set compact. The objective function is strictly increasing on \([0, \rho^i']\), thus the solution is uniquely defined by the largest \( \rho \) in the constraint set.

Proof of Proposition 4. The proof is based entirely on showing that if \( x_i > q^i(\omega) \), then \( G_i(p^i', \omega) \) is strictly increasing in \( \omega \). If \( x_i > q^i(\omega) \), then firm \( i \)'s gain from defection is:

\[
G_i(p^i', \omega) = \rho^i' q^i(\omega) - \rho^i \max \{ q^i(\omega)/2, q^i(\omega) - x_j \}.
\]

We show that for the case \( q^i(\omega)/2 > q^i(\omega) - x_j \) or \( q^i(\omega)/2 < q^i(\omega) - x_j \), the gain is strictly increasing. Since these functions are continuous, the maximum must be and the entire gain must be increasing.

First consider \( q^i(\omega)/2 > q^i(\omega) - x_j \), then

\[
G_i(p^i', \omega) = \rho^i' q^i(\omega)/2.
\]
Note that for this case \( \rho^l = \rho^m \). Taking the derivative of this expression with regards to \( \omega \), we have

\[
\frac{\partial G_i(\rho^m, \omega)}{\partial \omega} = \frac{1}{2} \frac{\partial \rho^m}{\partial \omega} \left( D(\rho^m, \omega) + \rho^m \frac{\partial D(\rho^m, \omega)}{\partial p} \right) + \left( \frac{\rho^m}{2} \right) \frac{\partial D(\rho^m, \omega)}{\partial \omega}.
\]

The first additive expression is zero based on the monopoly first order condition. Thus, the entire expression reduces to \( (\rho^m / 2) \frac{\partial D(\rho^m, \omega)}{\partial \omega} \), which is strictly positive based on Assumption 1.

Second consider the case that \( q^l(\omega)/2 < q^l(\omega) - x_j \), then

\[
G_i(\rho^l, \omega) = \rho^l x_j
\]

Note again that for this case \( \rho^l = \rho^m \). Now show this expression is strictly increasing in \( \omega \):

\[
\frac{\partial \rho^m}{\partial \omega} x_j = \left( -\rho^m \frac{\partial^2 D(\rho^m, \omega)}{\partial p \partial \omega} - \frac{\partial D(\rho^m, \omega)}{\partial \omega} \right) \frac{\partial D(\rho^m, \omega)}{\partial \omega} x_j.
\]

Based on Assumption 1, demand is concave and the denominator of the expression is negative. Assumption 2 tells us the numerator is also negative, and consequently the entire expression is positive.

**Proof of Proposition 5.** The condition \( x_2 \leq D(\tilde{p}^*, \tilde{\omega})/2 \) guarantees that the gain from defection for firm 2 is zero. Consequently, firm 1 must have the binding constraint in this state. Now consider SC pricing, which must satisfy the equality

\[
\rho^s \min \{D(\rho^s, \tilde{\omega}), x_1\} - \rho^s (D(\rho^s, \tilde{\omega}) - x_2) = L_1(p^s, \delta).
\]

Note for firm 2, \( G_2(\rho^s, \tilde{\omega}) = 0 < L_2(p^{**}, \delta) \).

Now consider the asymmetric prices \( \tilde{p}^*_1 \in (\rho^*, f_1^*(\rho^*)) \) and \( \tilde{p}^*_2 = \rho^s \). The period profit of firm 1 increases, since \( f_1^*(p^*_2) \) is defined as the residual maximizer. This makes the expected future loss from defection for firm 1 at least as large as was with symmetric pricing. The gain from a defection for firm 1 at these prices is

\[
\rho^s \min \{D(\rho^s, \tilde{\omega}), x_1\} - \tilde{p}^*_1 D_1^c(\tilde{p}^*_1, \rho^s, x_2, \tilde{\omega}),
\]

which is smaller than at symmetric pricing \( \rho^s \).

Note the expected loss from punishment of firm 2 is unchanged. Now consider the gain from defection of firm 2: \( G_2(\tilde{p}^*, \tilde{\omega}) = (\tilde{p}^*_1 - \rho^s) x_2 \), which becomes arbitrarily small for \( \tilde{p}^*_1 > \rho^s \) close enough to \( \rho^s \). Therefore, there is a \( \tilde{p}^*_1 > \rho^s \) such that \( G_2(\tilde{p}^*, \tilde{\omega}) \leq L_2(p^{**}, \delta) \).

Notice that joint profit is increased in state \( \tilde{\omega} \) since each firm 1 gets higher profits and
firm 2 has the same profit as with best symmetric pricing. Since the collusive prices in all states $\Omega \setminus \tilde{\omega}$ with pricing $\tilde{p}^s$ is also incentive compatible with pricing $\tilde{p}^*$, the discounted expected profits with colluding at $\tilde{p}^*$ in state $\tilde{\omega}$ is weakly larger, for both firms, than with the best symmetric pricing $\tilde{p}^s$.

Show that for any $\rho^s$ only firm 1 can have a binding constraint and always $\tilde{p}^s_j \geq \tilde{p}^s_i$, and thus $\tilde{\pi}^* \geq \tilde{\pi}^s$. Does not impact loss of firm 2 and increases loss of firm 1, thus it will allow weakly greater profit in all other periods. ■

We present a key lemma pertaining to symmetric pricing that is used in the proofs of Proposition 6 and 7.

**Lemma 1** For any $\epsilon > 0$ and $\tilde{\delta} - \epsilon, \pi(\tilde{p}^s, \tilde{\omega}) + \gamma(\epsilon) \leq \pi^J(\tilde{\omega})$ for some $\gamma(\epsilon) > 0$.

**Proof of Lemma 1.** Since $\tilde{p}^s$ is uniquely defined by $G_i(\tilde{p}^s, \tilde{\omega}) = L_i(\tilde{p}^s, \rho^s, \delta)$ and these functions are continuous symmetric prices and the discount factor, $\tilde{p}^s$ that must be continuous in $\delta$ close enough to $\tilde{\delta}$. We only consider $\delta$ close enough to $\tilde{\delta}$ that all other states are at JM pricing and firm $i$ is always the binding firms. By showing that $G_i(\tilde{p}, \tilde{\omega}) - L_i(\tilde{p}, \rho^s, \delta)$ is strictly increasing in $\tilde{p} \in [0, \rho']$ and strictly decreasing in $\delta \in (0, \tilde{\delta})$, we prove that a strictly smaller price makes the expression zero for a larger discount.

Clearly for fixed pricing $L_i(\tilde{p}, \rho^s, \delta)$ is strictly increasing in $\delta$ and $G_i(\tilde{p}, \tilde{\omega})$ is independent of $\delta$. Thus it only remains to show that $G_i(\tilde{p}, \tilde{\omega}) - L_i(\tilde{p}, \rho^s, \delta)$ is strictly increasing in $\tilde{p}$.

$$G_i(\tilde{p}, \tilde{\omega}) = \tilde{p} \max \{x_i, D(\tilde{p}, \tilde{\omega})\} - \pi_i(\tilde{p}, \tilde{\omega}),$$

and

$$L_i(\tilde{p}, \rho^s, \delta) = L_i^J(\delta) + \frac{\delta}{1 - \delta} \mu(\tilde{\omega}) (\pi_i(\tilde{p}, \tilde{\omega}) - \pi_i^J(\omega)).$$

Writing the entire expression $G_i(\tilde{p}, \tilde{\omega}) - L_i(\tilde{p}, \rho^s, \delta)$ we have:

$$\tilde{p} \max \{x_i, D(\tilde{p}, \tilde{\omega})\} - \pi_i(\tilde{p}, \tilde{\omega}) \left( \frac{\delta}{1 - \delta} \mu(\tilde{\omega}) + 1 \right) - L_i^J(\delta) - \frac{\delta}{1 - \delta} \mu(\tilde{\omega}) \pi_i^J(\omega).$$

The only pieces of this expression that change based on $\tilde{p}$ are (we write out $\pi_i(\tilde{p}, \tilde{\omega})$)

$$\tilde{p} \max \{x_i, D(\tilde{p}, \tilde{\omega})\} - \tilde{p} \max \{x_i, D(\tilde{p}, \tilde{\omega}) / 2, D(\tilde{p}, \tilde{\omega}) - x_i\} \left( \frac{\delta}{1 - \delta} \mu(\tilde{\omega}) + 1 \right). \quad (6)$$

We only need to consider prices such that the expression in (6) is positive, otherwise that price can never be feasible. First consider that case that $x_i < D(\tilde{p}, \tilde{\omega})$, then (6) reduces to $\tilde{p} (x_i - \max \{x_i, D(\tilde{p}, \tilde{\omega}) / 2, D(\tilde{p}, \tilde{\omega}) - x_i\} (\delta \mu(\tilde{\omega}) / (1 - \delta) + 1))$, which is clearly strictly increasing in $\tilde{p}$ on $[0, \rho']$ since the multiplicative terms in the expression $-\tilde{p}$ and
Lemma 1. This implies that to sustain joint collusive profits arbitrarily close to joint profit maximizing process. Using with more slack the closer constraints to hold at state positive compatibility. Condition 3 is exactly what is necessary for the incentive compatibility which are both strictly increasing expressions in \( \hat{p} \). Second consider the case that \( x_i \geq D(\hat{p}, \hat{\omega}) \). There are two subcases: (1) \( G_i(\hat{p}, \hat{\omega}) = (1 - \delta \mu(\hat{\omega})/(1 - \delta)) \hat{p} D(\hat{p}, \hat{\omega})/2 \) and (2) \( G_i(\hat{p}, \hat{\omega}) = \hat{p} x_j + \hat{p} (x_j - D(\hat{p}, \hat{\omega})) (\delta \mu(\hat{\omega})/(1 - \delta)) \), which are both strictly increasing expressions in \( \hat{p} \) on \([0, \hat{p}^l] \).

Proof of Proposition 6. Given condition 3 holds, we will verify that asymmetric collusion is incentive compatible for small enough \( \epsilon > 0 \) at \( \hat{\delta} - \epsilon \), and results in higher joint profit than the joint profit maximizing symmetric collusive pricing.

That is, take \( \hat{p}^*_j = \hat{p}^l \) and \( \hat{p}^*_i = \hat{p}^l - \eta \) for any arbitrarily small \( \eta > 0 \). The gain from defection of firm \( i \) in state \( \hat{\omega} \) becomes arbitrarily close to zero for small enough \( \eta > 0 \). The collusive profit of firm \( i \) must changes (from the symmetric pricing) for \( \hat{p}^l Q_i(\hat{p}^l, \hat{\omega}) \) to \((\hat{p}^l - \eta) x_i\), which is a strict increase otherwise the gain for firm \( i \) was zero and its constraint could not bind. This increases the loss from punishment for firm \( i \) and guarantees incentive compatibility. Condition 3 is exactly what is necessary for the incentive compatibility constraints to hold at state \( \hat{\omega} \) for at for sufficiently \( \epsilon > 0 \) and \( \eta > 0 \). The condition holds with more slack the closer \( \eta \) gets to zero. Thus, for small enough \( \epsilon > 0 \) it is always possible to sustain joint collusive profits arbitrarily close to joint profit maximizing process. Using Lemma 1 this implies that \( \lim_{\eta \to 0} \hat{\pi}^*_i(\hat{p}^l, \hat{p}^l - \eta) = \hat{\pi}^s > \hat{\pi}^l \), which guarantees this type of asymmetric pricing has higher joint profit than any incentive compatible symmetric pricing. For \( \epsilon \) small enough joint profit maximizing pricing is possible in all other states, which implies that for small enough \( \epsilon > 0 \), \( V^s(\hat{\delta} - \epsilon) > V^s(\hat{\delta} - \epsilon) \).

Proof of Proposition 7. The proof follows exactly the same lines as the proof of Proposition 6 with the pricing of \( \hat{p}^*_i = \hat{p}^l \) and \( \hat{p}^*_i = \hat{p}^l - \eta \) in state \( \hat{\omega} \) instead of \( \hat{\omega} \). Based on Condition 4 and Lemma 1, asymmetric collusion possible for small enough \( \epsilon > 0 \), and \( \hat{\delta} - \epsilon \), such that \( V^s(\hat{\delta} - \epsilon) > V^s(\hat{\delta} - \epsilon) \).

References


