Math 241 Sample Problems for Final Exam

Question 1 Let \( f(x, y) = \frac{\sin(2x - y)}{y} \).

a) Find the equation of the tangent plane to the surface \( f(x, y) \) at the point when \((x, y) = (2, 1)\).

b) Let \( z = g(x, y) \) and suppose that \( x(t) = t^2 + 3t + 2 \) and \( y(t) = e^t + \sin(3t) \). Find \( \frac{dz}{dt} \bigg|_{t=0} \) if

\[
\begin{align*}
\frac{\partial g}{\partial x} \bigg|_{(1,2)} &= 6, \quad \frac{\partial g}{\partial y} \bigg|_{(1,2)} = -2, \quad \frac{\partial g}{\partial x} \bigg|_{(2,1)} = -3, \quad \frac{\partial g}{\partial y} \bigg|_{(2,1)} = 8, \quad \frac{\partial g}{\partial x} \bigg|_{(0,0)} = 0, \quad \frac{\partial g}{\partial y} \bigg|_{(0,0)} = -4
\end{align*}
\]

Question 2 Let the temperature at a point \((x, y)\) be given by

\[ T(x, y) = xy(1 + x^2 + 2y^2) \]

a) Find the direction in which the temperature rises most rapidly at \((1, 2)\).

b) Find the directional derivative of \( T \) at the point \((1, 2)\) in the direction of the vector \( \mathbf{v} = 5 \mathbf{i} - \mathbf{j} \).

Question 3 Let \( f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2 \).

a) Find the critical points of \( f(x, y) \).

b) Classify the critical points in part a) as a relative maximum, relative minimum or saddle point.

Question 4 Find the volume of the solid wedge cut from the cylinder \( 4x^2 + y^2 = 16 \) below by the plane \( z = 0 \) and above by the plane \( z = y \) by evaluating an appropriate double integral.

Question 5 Evaluate the double integral

\[
\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{(1 + x^2 + y^2)^{3/2}} \, dx \, dy,
\]

by using polar coordinates.

Question 6 Express the triple integral:

\[
\iiint_R \frac{1}{x^2 + y^2 + z^2} \, dy \, dz \, dx
\]

as an integral in spherical coordinates if \( R \) is the region bounded below by the paraboloid \( 2z = x^2 + y^2 \), and above by the sphere \( x^2 + y^2 + z^2 = 8 \). This is a little tricky since you will need to use two triple integrals. DO NOT Evaluate the integrals!

Question 7 Let \( \mathbf{F}(x, y) = (e^x \sin y - y) \mathbf{i} + (e^x \cos y - x - 2) \mathbf{j} \) be a vector field defined on \( \mathbb{R}^2 \).

a) Show that \( \mathbf{F} \) is a conservative vector field.

b) Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the path \( \mathbf{r}(t) = (\ln(t + 1) \cos(\sqrt{\pi} \, t)) \mathbf{i} + (t^2 + \frac{1}{2} \pi) \mathbf{j} \), \( 0 \leq t \leq \frac{\sqrt{\pi}}{2} \).

Question 8 Evaluate the line integral

\[
\int_C (x + xy^2) \, dx + 2(x^2y - y^2 \sin y) \, dy
\]

where \( C \) is the path oriented counterclockwise enclosing the region in the first quadrant bounded by \( y = x^2 \) and \( y = 1 \) and \( x = 0 \) by using Green’s Theorem.

Question 9 Use the transformation \( x = u^{2/3} v^{1/3}, y = u^{1/3} v^{2/3} \) to find

\[
\iint_R \frac{x^2 \sin xy}{y} \, dA
\]

where \( R \) is the triangular region bounded by the parabolas \( x^2 = \frac{1}{2} \pi y, x^2 = \pi y, y^2 = \frac{1}{4} x, y^2 = x \).

Question 10 Something from 17.5 or later?