Math 241 Sample Problems for Final Exam

Question 1 Let \( f(x, y) = \frac{\sin(2x - y)}{y} \).

a) Find the equation of the tangent plane to the surface \( f(x, y) \) at the point when \((x, y) = (2, 1)\).

b) Let \( z = g(x, y) \) and suppose that \( x(t) = t^2 + 3t + 2 \) and \( y(t) = e^t + \sin(3t) \). Find \( \frac{dz}{dt} \) if \( \frac{\partial g}{\partial x} \bigg|_{(1,2)} = 6, \frac{\partial g}{\partial y} \bigg|_{(1,2)} = -2, \frac{\partial g}{\partial x} \bigg|_{(2,1)} = -3, \frac{\partial g}{\partial y} \bigg|_{(2,1)} = 8, \frac{\partial g}{\partial x} \bigg|_{(0,0)} = 0, \frac{\partial g}{\partial y} \bigg|_{(0,0)} = -4 \).

Question 2 Let the temperature at a point \((x, y)\) be given by \( T(x, y) = xy(1 + x^2 + 2y^2) \).

a) Find the direction in which the temperature rises most rapidly at \((1, 2)\).

b) Find the directional derivative of \( T\) at the point \((1, 2)\) in the direction of the vector \( \mathbf{v} = 5\mathbf{i} - \mathbf{j} \).

Question 3 Let \( f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2 \).

a) Find the critical points of \( f(x, y) \).

b) Classify the critical points in part a) as a relative maximum, relative minimum or saddle point.

Question 4 Find the volume of the solid wedge cut from the cylinder \( 4x^2 + y^2 = 16 \) below by the plane \( z = 0 \) and above by the plane \( z = y \) by evaluating an appropriate double integral.

Question 5 Evaluate the double integral \( \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{(1 + x^2 + y^2)^{3/2}} \, dx \, dy \), by using polar coordinates.

Question 6 Express the triple integral: \( \iiint_R \frac{1}{x^2 + y^2 + z^2} \, dy \, dz \, dx \) as an integral in spherical coordinates if \( R \) is the region bounded below by the paraboloid \( 2z = x^2 + y^2 \), and above by the sphere \( x^2 + y^2 + z^2 = 8 \). This is a little tricky since you will need to use two triple integrals. DO NOT Evaluate the integrals!

Question 7 Let \( \mathbf{F}(x, y) = (e^x \sin y - y)\mathbf{i} + (e^x \cos y - x - 2)\mathbf{j} \) be a vector field defined on \( \mathbb{R}^2 \).

a) Show that \( \mathbf{F} \) is a conservative vector field.

b) Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the path \( \mathbf{r}(t) = (\ln(t + 1) \cos(\sqrt{\pi} \, t))\mathbf{i} + (t^2 + \frac{1}{2} \pi)\mathbf{j}, 0 \leq t \leq \frac{\sqrt{\pi}}{2} \).

Question 8 Evaluate the line integral \( \int_C (x + xy^2) \, dx + 2(x^2y - y^2 \sin y) \, dy \) where \( C \) is the path oriented counterclockwise enclosing the region in the first quadrant bounded by \( y = x^2 \) and \( y = 1 \) and \( x = 0 \) by using Green’s Theorem.

Question 9 Use the transformation \( x = u^{2/3}v^{1/3}, y = u^{1/3}v^{2/3} \) to find
\[
\iint_R \frac{x^2 \sin xy}{y} \, dA
\]
where \( R \) is the triangular region bounded by the parabolas \( x^2 = \frac{1}{2} \pi y, x^2 = \pi y, y^2 = \frac{1}{4} x \).

Question 10 Something from 17.5 or later?