Math 404 Graded Homework 2
Name:__________________________
Due April 29, 2003

To receive full credit, you must show all work.

**Question 1** This is exactly problem 11 from section 2.2 in the book. Prove that a straight line is the shortest curve that joins two points in \( \mathbb{R}^3 \). Do this the following way: Let \( c : [a, b] \rightarrow \mathbb{R}^3 \) be an arbitrary curve from \( p = c(a) \) to \( q = c(b) \). Let \( u = (q - p)/\|q - p\| \).

a) Show that if \( \sigma \) is a straight line segment from \( p \) to \( q \), say \( \sigma(t) = (1 - t)p + tq, 0 \leq t \leq 1 \), then \( L(\sigma) = d(p, q) \).

b) Cauchy-Schwartz implies that \( \|c'\| \geq c' \cdot u \). Use this to deduce that \( L(c) \geq d(p, q) \).

c) Show that if \( L(c) = d(p, q) \), then \( c \) is a straight line segment.

**Question 2** Now we are going to investigate the same problem using the calculus of variations. Very often in math or physics, one is interested in minimizing or maximizing a functional. For our purposes a functional \( F \) will be a function from some set of functions to \( \mathbb{R} \). These are often given by integrals. For example, consider the set \( C \) of all smooth curves \( c \) in the plane joining \( p \) and \( q \) and parametrized on the interval \( [a, b] \). Then the length functional \( L \) is

\[
L(c) = \int_a^b \|c'\| \, dt
\]

If we further assume that \( c \) is the graph of a function \( y = c(t) \) joining the points \( p = (a, c(a)) \) to \( q = (b, c(b)) \), then \( L \) can be written as

\[
L(c) = \int_a^b \sqrt{1 + (c')^2} \, dt
\]

To find the shortest curve joining \( p \) to \( q \), we would like to “differentiate \( L \) with respect to \( c \)” and set the result equal to 0 to find the “critical curves” which we hope are minimums or shortest curves (geodesics).

Here is the general framework in which to do this. Consider a suitably differentiable function \( F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), given by \( F(t, x, y) \). We wish to find the maxima/minima of the functional

\[
J(c) = \int_a^b F(t, c(t), c'(t)) \, dt
\]

(To get the length functional, let \( F = \sqrt{1 + y'^2} \).)

Now we consider a variation of \( c \) with endpoints fixed, that is, a function

\[
\alpha : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow \mathbb{R}
\]

such that \( \alpha(0, t) = c(t) \) and \( \alpha(u, a) = p \) and \( \alpha(u, b) = q \) for all \( u \in (-\varepsilon, \varepsilon) \). Note that for fixed \( u = u_0 \), \( \alpha(u_0, t) \) is just a curve joining \( p \) to \( q \). See the picture. As \( u \) varies we get a family of curves which “pass through” \( c \) when \( u = 0 \). Denote the \( u \)-th curve by \( \alpha_u(t) \).
a) Now it’s your turn to do some stuff. For a variation $\alpha$, show that

$$
\frac{d}{du} \left( J(\alpha(u)) \right) \bigg|_{u=0} = \frac{d}{du} \left( \int_a^b F(t, \alpha(u, t), \frac{\partial \alpha}{\partial t}(u, t)) \, dt \right) \bigg|_{u=0}
$$

$$
= \int_a^b \left[ \frac{\partial \alpha}{\partial u}(0, t) \frac{\partial F}{\partial x}(t, c(t), c'(t)) + \frac{\partial^2 \alpha}{\partial u \partial t}(0, t) \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right] \, dt
$$

Since mixed partials are equal, $\frac{\partial^2 \alpha}{\partial u \partial t} = \frac{\partial^2 \alpha}{\partial t \partial u}$, apply integration by parts to the second term in the integrand and use the fact that endpoints are fixed to conclude

$$
\frac{d}{du} \left( J(\alpha(u)) \right) \bigg|_{u=0} = \int_a^b \frac{\partial \alpha}{\partial u}(0, t) \left[ \frac{\partial F}{\partial x}(t, c(t), c'(t)) - \frac{d}{dt} \left( \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right) \right] \, dt
$$

b) Thus critical points of $J$ correspond to curves $c$ with

$$
\frac{\partial F}{\partial x}(t, c(t), c'(t)) - \frac{d}{dt} \left( \frac{\partial F}{\partial y}(t, c(t), c'(t)) \right) = 0
$$

This is called the Euler-Lagrange equation of the functional $J$. Use this to show that straight lines are critical points of the length functional $L$. ($F(t, x, y) = \sqrt{1 + y'^2}$.) To show these are actually minima we would have to compute the second derivative of $J$ with respect to $u$ and use the second derivative test. This can be done, but is a big mess!

c) Suppose now that you wanted to find a curve $c$ given as a graph $y = c(t)$ over $[a, b]$, for which the surface of revolution obtained by rotating $c$ about the $t$–axis has minimal area amongst all curves joining $(a, c(a))$ to $(b, c(b))$. To make the problem interesting we assume that $c(t) > 0$ on $[a, b]$. This will give a so-called minimal surface of revolution. What should the function $F$ be, so that the corresponding functional $J$ represents the area of the surface of revolution? Deduce that a curve $c$ that generates a minimal surface of revolution satisfies the non-linear differential equation

$$
1 + \left( \frac{dc}{dt} \right)^2 - c(t) \left( \frac{d^2 c}{dt^2} \right) = 0
$$

Miraculously, this differential equation can be solved since the independent variable $t$ is missing using some standard tricks. See, for example, the Boyce–DiPrima book on differential equations. It turns out that the solution to this differential equation is $c(t) = C \cosh \left( \frac{t + K}{C} \right)$, where $C$ and $K$ are constants. The resulting surfaces are called catenoids.