In giving your answer, the answer alone is not enough. Make sure you clearly give your rationale for arriving at the answer. It must be clear to me how you arrive at your answer.

Points: a) 25% b) 15% c) 10% d) 20% e) 10%

1. A control loop consists of a controller \((K_p)\) and a second-order plant \((K_2, \omega_n, \zeta)\). It has a first-order actuator \((K_i, T)\). The second-order plant puts out a velocity that is then integrated up to a displacement. The control-loop is a unity-feedback loop.

   a. Draw this loop.

   ![Control Loop Diagram]

   b. What is the loop's open-loop transfer function?

   \[
   G_{OL} = \frac{K_p K_i K_2 \omega_n^2}{(Ts+1)(s^2 + 2\zeta \omega_n s + \omega_n^2)}
   \]

   c. What is the order of this \(G_{OL}\)?

   3rd order \((\frac{1}{s} \text{ has no effect})\)

   d. Figure out the ODE that corresponds to the transfer function \(G_{OL} = \frac{x}{x_r}\)

   \[
   \frac{x}{x_r} = \frac{K_p K_i K_2 \omega_n^2}{[Ts^4 + (1 + 2\zeta \omega_n T)s^3 + (2\zeta \omega_n + Tw_n)s^2 + \omega_n^2]} \leq 0
   \]

   \[
   T \dddot{x} + (1 + 2\zeta \omega_n T) \ddot{x} + (2\zeta \omega_n + Tw_n) \dot{x} + \omega_n^2 x = K_p K_i K_2 \omega_n^2 x_r
   \]

   e. What is the closed-loop transfer function for this system?

   \[
   G_{CL} = \frac{N_{cl}}{D_{cl} + N_{cl}} = \frac{K_p K_i K_2 \omega_n^2}{[Ts+1](s^2 + 2\zeta \omega_n s + \omega_n^2) + K_p K_i K_2 \omega_n^2}
   \]
f. What is the order of the closed-loop system?

4th, there is a \( s^4 \) term in denominator

g. What is the closed-loop steady-state gain?

\[
K_{CL} = \frac{K_p K_1 K_2 w_n^2}{K_p K_1 K_2 w_n^2} = \frac{K_p K_1 K_2 w_n^2}{K_p K_1 K_2 w_n^2} = 1
\]

(divide numerator by \( s^0 \) coefficient in denominator)

h. Redraw the loop with velocity control instead of position control, using the same controller, actuator, and plant.

![Redrawn loop diagram]

i. What is the closed-loop system's transfer function \( \frac{v}{v_c} \)?

\[
G_{CL} = \frac{K_p K_1 K_2 w_n^2}{(T_s + 1) (s^2 + 2 \delta w_n s + w_n^2) + K_p K_1 K_2 w_n^2}
\]

j. What is the closed-loop steady-state gain for this velocity control loop?

The \( s^0 \) coefficient here is \( w_n^2 (1 + K_p K_1 K_2) \), so

\[
K_{CL} = \frac{K_p K_1 K_2}{(1 + K_p K_1 K_2)}
\]

k. What is this closed-loop's order?

3rd

l. Now remove the P-only controller and put in a PD controller with \( K_p = 1 \) and \( K_D = T \).

Draw this controller and get its transfer function.

![PD controller diagram]

\[
G_{PD} = 1 + T s
\]

m. What is the closed-loop transfer function of this PD-loop?

\[
G_{CL} = \frac{(1 + Ts) K_1 K_2 w_n^2}{(T_s + 1) (s^2 + 2 \delta w_n s + w_n^2) + (1 + Ts) K_1 K_2 w_n^2}
\]

\[
G_{CL} = \frac{K_1 K_2 w_n^2}{s^2 + 2 \delta w_n s + w_n^2 + K_1 K_2 w_n^2}
\]
n. What is the PD-loop's closed-loop steady-state gain?

\[
K_{cl} = \frac{K_1 K_2 \omega_n}{\omega_n^2 (1 + K_1 K_2)} = \frac{K_1 K_2}{1 + K_1 K_2}
\]

o. What is its closed-loop natural frequency?

\[
\omega_{n-cl} = \omega_n \sqrt{1 + K_1 K_2}
\]

p. What is its closed-loop damping ratio?

\[
\zeta_{cl} = \frac{5 \omega_n - \omega_n}{\omega_n} = \frac{5 \omega_n}{\omega_n \sqrt{1 + K_1 K_2}} = \frac{5}{\sqrt{1 + K_1 K_2}}
\]

q. If \(K_1 = K_2\) and \(\zeta = \frac{1}{2}\), what are the values of \(K_1\) and \(K_2\)?

The plant itself is critically damped and

\[
\zeta_{cl} = \frac{1}{2} = \frac{1}{\sqrt{1 + K_1^2}} \quad \sqrt{1 + K_1^2} = 2
\]

\[K_1^2 = 4 - 1 = 3\]

\[K_1 = K_2 = \sqrt{3}\]

5. \(G \begin{array}{c} G_{cl} = \frac{G}{1+GH} \end{array} \begin{array}{c} G_1 = \frac{K_{ss}}{T_s+1} \end{array} \begin{array}{c} G_2 = \frac{K_{ss} \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{K_{ss}}{\omega_n^2 + 2\zeta \omega_n s + 1} \end{array} \)

\(a - p \quad 160 \quad \frac{100}{16} \quad 6 \text{ on avg}\)