9.4 Vector approach to rigid-body kinematic analysis of velocities

There are several approaches to analyzing 2-D rigid-body kinematic problems. They can be solved simply trigonometrically, for example, applying the rules of trigonometry to a mechanism, noting geometric features of it. Another approach is a more formal, rigid vector approach. This approach is explained in this section. For beginners in rigid-body motion, it offers a structure that is set and can be applied to all 2-D kinematic problems involving rigid bodies.

The key to this approach is to pick two points on a rigid body whose velocities are known or are partially known. In many cases the magnitude of the velocity, the speed, of a point is not known, but it is constrained to move in a certain direction. Let’s say that there is a rigid body with two points on it, $A$ and $B$, about which the velocities are known or partially known. The relative velocity relationship can be written relating the velocities of these two points.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

Note that the subscripts in this equation follow the pattern $B$-$A$-$B$-$A$. This is always the case. It would be equally valid to write the $A$-$B$-$A$-$B$ equation, and in most problems it does not matter which one is written. Both are valid. From this equation, we can expand the term $\vec{v}_{A/B}$.

$$\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B}$$

These two equations form the basis of this vector approach and are employed sequentially over and over as we work our way through the mechanism, applying this equation in stages, to pairs of points on different components of the mechanism. To show how this is done, let’s look at a classical mechanism in rigid-body kinematics.

Example 9.*** - Slider/Crank mechanism

At right is shown a slider/crank mechanism, which is the basis for converting linear motion into circular motion in a reciprocating engine or compressor. The piston $B$ runs vertically up and down in the cylinder. The link $AB$ is the connecting rod. $OA$ is the crank or crank arm. It rotates about the fixed crank center at $O$. For this discussion let’s assume that $\omega_{OA}$ is constant and is given. The lengths of the links—$\ell_{OA}$ and $\ell_{AB}$—are also given, as are $\theta$ and $\phi$ in this position. We are interested in the angular velocity of $AB$ and the speed of the piston in this configuration.

We start with what is known ( $\vec{\omega}_{OA}$ ) and work our way toward what is unknown ( $\vec{v}_B$ ).

We can get the velocity of $A$, knowing $\vec{\omega}_{OA}$.
\[ \vec{v}_A = \vec{\omega}_{OA} \times \vec{r}_{A/O} \]

The rotational velocity of OA is given \( \vec{\omega}_{OA} = \omega_{OA} \hat{k} \), and \( \vec{r}_{A/O} = l_{OA} (\cos \theta \hat{i} + \sin \theta \hat{j}) \). Thus

\[ \vec{v}_A = \omega_{OA} \hat{k} \times l_{OA} (\cos \theta \hat{i} + \sin \theta \hat{j}) \]

We can pull the two scalars out front.

\[ \vec{v}_A = \omega_{OA} l_{OA} \hat{k} \times (\cos \theta \hat{i} + \sin \theta \hat{j}) \]

Now we use the methodology explained in section ***.*** to perform the cross product.

\[ \vec{v}_A = \omega_{OA} l_{OA} (\cos \theta \hat{j} - \sin \theta \hat{i}) \]

This implies that A points to the left and up, which is certainly what we would expect. Now we turn our attention to link AB. With \( \vec{v}_A \) known, we are interested in the velocity of point B. Relate the two points.

\[ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \]
\[ \vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B} \]

To an observer on B looking down at A, A looks as if it is moving to the right. Thus the direction of rotation of link AB is clockwise—i.e., \( \vec{\omega}_{AB} = \omega_{AB} (-\hat{k}) \). \( \vec{r}_{A/B} = l_{AB} (\cos \phi \hat{i} - \sin \phi \hat{j}) \). Thus

\[ \vec{v}_{A/B} = \omega_{AB} (-\hat{k}) \times l_{AB} (\cos \phi \hat{i} - \sin \phi \hat{j}) \]
\[ \vec{v}_{A/B} = \omega_{AB} l_{AB} (-\hat{k}) \times (\cos \phi \hat{i} - \sin \phi \hat{j}) = \omega_{AB} l_{AB} (\cos \phi \hat{j} + \sin \phi \hat{i}) \]

The piston is moving upward in the cylinder at this point, so \( \vec{v}_B = v_B \hat{j} \). Putting this all together

\[ v_B \hat{j} = \omega_{OA} l_{OA} (\cos \theta \hat{j} - \sin \theta \hat{i}) + \omega_{AB} l_{AB} (\cos \phi \hat{j} + \sin \phi \hat{i}) \]

This single, 2-D vector equation thus yields two scalar equations.

\[ \hat{i} : 0 = -\omega_{OA} l_{OA} \sin \theta + \omega_{AB} l_{AB} \sin \phi \]
\[ \hat{j} : v_B = \omega_{OA} l_{OA} \cos \theta + \omega_{AB} l_{AB} \cos \phi \]

There are two unknowns in these two equations, \( \omega_{AB} \) and \( v_B \), which are the two unknowns sought.

Note that the \( \hat{i} \)-equation simply states the fact that the \( \hat{i} \)-component of \( \vec{v}_A \) as calculated from the rotational motion of OA is the same as the \( \hat{i} \)-component of \( \vec{v}_A \) as calculated from the rotational motion of AB.

### 9.5 Rigid-body velocity analysis using velocity diagrams
A more intuitive, graphical approach is available for this analysis too. It uses *velocity diagrams* and the fact that no two points on a rigid body can approach or get further away from each other. The body is rigid after all.

**Example 9.*** — Slider/Crank mechanism via velocity diagram

At left is shown a stick figure of the slider/crank mechanism. Again, $\omega_{OA}$ is given as well as the dimensions of the links and the angles $\theta$ and $\phi$, as in Example 9.*** The right-hand drawing is of link $AB$, the link whose motion is unknown. $\vec{v}_A$ is found as before. With $\vec{v}_A$ known, it is possible through trigonometry to find $\omega_{AB}$ and $v_B$. Here’s how.

Since link $AB$ is rigid, points $A$ and $B$ must have the same component of velocity along the link. We can resolve $\vec{v}_A$ and $\vec{v}_B$ into components parallel ($\parallel$) and perpendicular ($\perp$) to the link.

**Figure 9.11 — Slider/Crank velocity diagram**

It must be so that

$$\vec{v}_{A\parallel} = \vec{v}_{B\parallel}$$

We can write the trigonometric relationships as illustrated in the figure.

1. $\theta$ is the original angle given for $OA$
2. $\psi = 90^\circ - \theta$
3. This angle is $\psi$ because it’s the opposite interior angle of 2
4. $\phi$ is the original angle given for $AB$
5. This is the same $\phi$ between the horizontal and $AB$
6. $\beta = 90^\circ - \phi$

Thus $\psi$ and $\beta$ can be found. Knowing $v_A$

$$v_{A\parallel} = v_{B\parallel}$$

$$v_A \cos(\phi - \psi) = v_B \cos(\beta)$$

$$v_B = \frac{v_A \cos(\phi - \psi)}{\cos(\beta)}$$

With $v_B$, we can find $v_{B\perp}$. If you were standing on $A$ looking at $B$, $B$ would look as if it were going to the right. But you would have velocity $v_{A\parallel}$ to the left. So the apparent sideways speed of $B$ viewed from $A$ would be $v_{A\perp} + v_{B\perp}$ to the right. Thus
This last equation is derived from the well-known relationship for relative velocities

\[ \vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B} \]

Other approaches are shown in the figure below. These shows two possible velocity diagrams that can be used to find \( \vec{v}_B \) and \( \vec{v}_{B/A} \) from \( \vec{v}_A \).

**Figure 9.12 – Velocity diagrams for slider/crank mechanism**

In the left-hand diagram, \( \vec{v}_A \) is drawn, as well as a line through its tail that is along \( AB \). This allows \( \vec{v}_A \) and \( \vec{v}_B \) are placed tail to tail. This allows \( \vec{v}_{A\parallel} \) to be projected onto that line. Of course \( \vec{v}_{A\parallel} = \vec{v}_{B\parallel} \). We then draw a vertical line through the tail of \( \vec{v}_A \), since this is the direction of \( \vec{v}_B \). This allows us to draw \( \vec{v}_B \), since \( \vec{v}_{B\parallel} \) is just its projection. We can then draw a line perpendicular to the direction of \( AB \) through the tail of \( \vec{v}_A \). This allows us to project \( \vec{v}_{A\perp} \) and \( \vec{v}_{B\perp} \) onto this line. With this diagram we can do the trigonometry as follows.

1. Draw \( \theta \) on the diagram
2. \( \psi = 90^\circ - \theta \)
3. \( \phi \) is the original angle given for \( AB \)
4. \( \beta = 90^\circ - \phi \)
5. An inspection of the right angles in the figure shows that this angle is \( \beta \) also

From this, it can be seen that

\[ v_{A\parallel} = v_A \cos(\phi - \psi) \]

\[ v_B = \frac{v_{A\parallel}}{\cos(\beta)} \]

\[ \vec{v}_{A\perp} = v_A \sin(\phi - \psi), \quad \vec{v}_{A\perp} = v_B \sin(\beta) \]
In the diagram on the right, a different approach is taken. It is based on the equation

\[ \vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \]

The only vector known at the outset is \( \vec{v}_A \). \( -\vec{v}_A \) is drawn. A vertical line is drawn through its head, which represents the direction of \( \vec{v}_B \). The direction of link \( AB \) is drawn, as well as a line perpendicular to this direction. \( \vec{v}_{B/A} \) must lie on this perpendicular line, since point \( A \) doesn’t approach or get farther away from point \( B \). With the vertical line and the line perpendicular to \( AB \), it is possible to find the vectors \( \vec{v}_B \) and \( \vec{v}_{B/A} \), after doing a little angle trigonometry. That follows as

1. Draw \( \theta \) on the diagram
2. \( \psi = 90^\circ - \theta \), since \( \theta + 90^\circ + \psi = 180^\circ \)
3. \( \phi \) is the original angle given for \( AB \)
4. \( \beta = 90^\circ - \phi \)
5. An inspection of the right angles in the figure shows that this angle is \( \beta \) also

From this diagram

\[ v_{B/A} \cos(\beta) = v_A \cos(\psi), \quad v_{B/A} = v_A \frac{\cos(\psi)}{\cos(\beta)} \]

\[ \omega_{AB} = \frac{v_{B/A}}{l_{AB}} \]

\[ v_B = v_A \sin(\psi) + v_{B/A} \sin(\beta) \]

There are certainly other diagrams, akin to these, that could be drawn to represent the equation relating \( \vec{v}_A \) and \( \vec{v}_B \). These are just some possibilities.

As can be seen, this approach is more hands-on and graphical. It can be implemented in multiple ways. It involves more trigonometry too. It also requires a different approach with each mechanism, because each mechanism’s geometry is different. Both approaches—the vector approach and the graphical approach—have their advantages and disadvantages. Which to use is up to the dynamicist. Both work. A mixture of the two is also useful.