Example 10.*** – Rolling wheel

A rolling wheel is a good place to start for understanding rigid-body kinematics because it contains several surprises that are counter-intuitive. Figure 10.3 shows a rolling wheel connected to a vehicle at point $O$ that is moving at a constant velocity $v_O$ to the right. Thus the wheel is rotating clockwise. There is no slipping between the wheel and the surface that it is travelling along. That is point $A$ has no velocity, since it is not moving relative to the surface at the instant shown. We shall consider the motion of the wheel at the instant shown to be able to talk about the velocities of various points on the wheel that are in the locations (instantaneously) demarcated by $A$, $B$, $C$, $D$, and $O$.

The first surprise (perhaps) is that the wheel is rotating about point $A$ and not point $O$. Sure, the axle is at point $O$, but the vehicle has motion, so point $O$ is moving. Point $A$ has no motion because it has no motion relative to its contact point on the ground.

If the velocity of the vehicle is known, we can calculate the rotational velocity of the wheel ($\omega_{AO}$).

$$\vec{v}_O = \vec{\omega}_{AO} \times \vec{r}_{O/A} = \omega_{AO} (\vec{k}) \times r_{AO} \hat{j}$$

$$v_O \hat{i} = \omega_{AO} r_{AO} \hat{j}$$

$$\omega_{AO} = \frac{v_O}{r_{AO}}$$

The formality of the vector algebra is shown here. The use of the unit vectors and the subtle manipulation of the quantities’ subscripts is important to understand. For example, $\vec{r}_{O/A}$ is freighted with meaning, with which point is the reference point—where the observer stands and where he/she directs his/her vision—whereas $r_{AO}$ is just a scalar value, the distance between the points $A$ and $O$, without reference to any particular point or points. The $OA$ in the subscript just refers to the body of which this is the radius. Note that the vector algebra has to be worked out so that there are vectors pointed in the same direction ($\hat{i}$) on each side of the equation (the second equation), at which point we can extract a scalar equation from the vector equation. Arriving at a scalar equation allows us to perform division, something not allowed with vector quantities.

A similar calculation can be performed for point $C$:
\[ \vec{v}_C = \vec{\omega}_{AO} \times \vec{r}_{C/A} = \omega_{AO}(-\hat{k}) \times r_{AC}\hat{j} \]

\[ v_C \hat{i} = \omega_{AO}r_{AC} \hat{i} \]

\[ v_C = \omega_{AO}r_{AC} \]

This shows that \( v_C \) is twice the value of \( v_O \), since \( r_{AC} = 2r_{AO} \). Note also that we could have changed the subscript of \( \omega \) from \( AO \) to \( AC \), but this is unnecessary and perhaps confusing. The body has only one rotation rate, so that

\[ \vec{\omega}_{AO} = \vec{\omega}_{AC} = \vec{\omega}_{BC} = \ldots \]

Let’s now look at the velocity of point \( B \).

\[ \vec{v}_B = \vec{\omega}_{AO} \times \vec{r}_{B/A} = \omega_{AO}(-\hat{k}) \times [r(-\hat{i}) + r\hat{j}] = \omega_{AO}r(-\hat{k}) \times [(-\hat{i}) + \hat{j}] \]

\[ (-\hat{k}) \times (-\hat{i}) = \hat{k} \times \hat{i} = \hat{j} \]

since the permutation \( \hat{i} \rightarrow \hat{j} \rightarrow \hat{k} \rightarrow \hat{i} \rightarrow \ldots \) is positive.

\[ (-\hat{k}) \times (\hat{j}) = \hat{i} \]

since this is a negative permutation, but \( \hat{k} \) has a minus sign too. Thus

\[ \vec{v}_B = \omega_{AO}(\hat{i} + \hat{j}) \]

that is, point \( B \) is moving upward to the right. The magnitude of \( v_B \) is

\[ v_B = \sqrt{2}\omega_{AO}r = \sqrt{2}\omega_{AO}r_{AO} = \sqrt{2}v_O \]

since the length of \( (\hat{i} + \hat{j}) \) is \( \sqrt{2} \). We can make a general observation here. Since \( A \) is the center of rotation of the wheel at any instant and the wheel is a rigid body, the movement of any point on the wheel is perpendicular to a line between that point and point \( A \). The speed of the point is proportional to its distance from point \( A \). Using this we can say then that the speed of point \( D \) is the same as the speed of point \( B \), since \( B \) and \( D \) are the same distance from point \( A \). But the velocity of point \( D \) will point to the right and down. This can be confirmed by performing the vector calculation illustrated for point \( B \) on point \( C \).

The accelerations of the points on the wheel are a bit more complicated. This is because, unlike with velocity, there is normal acceleration between points on a rigid body if the body is rotating. The acceleration of any point is equal to its acceleration relative to a second point plus the acceleration of that second point. Let’s talk about specific points, \( A \) and \( O \), to make this clear.

The only point on the wheel that is not accelerating is point \( O \). It is moving in a straight line at a constant speed. Thus there is no speed-up acceleration \( (\vec{a}_s) \) and no change-of-direction acceleration \( (\vec{a}_n) \). So \( \vec{a}_{A/O} = \vec{a}_A \) since \( \vec{a}_O = 0 \). This is true of all points on the wheel, not just point \( A \). Now
\[ \ddot{a}_{A/O} = \ddot{a}_{A/O-n} + \ddot{a}_{A/O-t} = \ddot{a}_{A/O-n} \]

There is no tangential acceleration because the wheel’s rotational velocity is constant.

\[
\ddot{a}_{A/O-n} = \ddot{\omega}_{AO} \times (\dddot{\omega}_{AO} \times \ddot{r}_{A/O}) = \omega_{AO} (-\hat{k}) \times [\omega_{AO} (-\hat{k}) \times r(-j)] \\
= \omega_{AO}^2 r(-\hat{k}) \times [(-\hat{k}) \times (-j)] = \omega_{AO}^2 r(-\hat{k}) \times (-i) = \omega_{AO}^2 r \hat{j}
\]

Thus

\[
\ddot{a}_A = \ddot{a}_{A/O} = \ddot{a}_{A/O-n} = \omega_{AO}^2 r \hat{j}
\]

Point A is accelerating upward toward point O.

The same analysis would show that point C is accelerating downward toward point O but has no tangential acceleration, since \( \ddot{a}_{C/O} = \ddot{a}_{AO} \times \dddot{r}_{C/O} \), and since the wheel is rolling with a constant speed, \( \ddot{a}_{AO} = 0 \). This seems paradoxical because at A the point has no velocity, yet at C, the point is moving at a speed of \( 2v_0 \). How then did this point on the rim speed up? The answer to this is that each point actually follows a curve called a cycloid. The figure below shows the path followed by a point at A as the wheel rolls. The points B, C, and D are shown along this path. Note that at each position along this path the overall acceleration of the point is toward point O. At A \( \ddot{a}_A \) is tangential upward. The speed of the point along the path as it approaches A decreases, reaches O at A, then begins to increase between A and B. At B \( \ddot{a}_{Bt} \) is still pointed in the direction of motion, so \( \dot{v} \) is still increasing at B. Notice at C \( \ddot{a}_C \) is straight down toward O. Without a tangential component of acceleration, \( \dot{v} \) is neither increasing nor decreasing. This is the only point on the path where \( \dot{v} \) is not changing. So from A to C along the cycloid, the speed of the point is increasing, but ever more slowly. After C the tangential component of \( \ddot{a} \) is pointed backwards from the direction of travel, as can be seen at D. The speed is decreasing from its maximum at C to 0 again at A between C and A. Another subtlety is that, though at C \( \dot{v} \) is not changing, \( \ddot{v} \) is changing greatly. Its direction, \( \dot{\ddot{v}} \), is changing at its maximum rate. In fact, by comparing \( \ddot{a}_A \) and \( \ddot{a}_C \), one can see that at A, the maximum speed-up is at A, where \( \ddot{a}_A \) is tangent to the path, and the maximum direction-change is at C, where \( \ddot{a}_C \) is normal to the path.

![Cycloidal motion of point on rim of rolling wheel](image)

*Figure 10.4 – Cycloidal motion of point on rim of rolling wheel*
It is quite a surprise to see how complicated the kinematics can be for a simple, one-part mechanism like a rolling wheel. It’s surprising how much there is to be learned from this simple device. And so far, we’ve only looked at the motion of points on a wheel rolling at a constant speed.