Let’s explore projectile motion a bit more. For the following analysis, ignore air resistance and any variations in the Earth’s gravitational field. Answer the following questions.

a) Assume you know the initial velocity with which a projectile is launched ($v_0$). What launch angle ($\theta_0$) will give the greatest range, if the landing point is at the same level as the launching point? Find your answer in terms of $v_0$ and $g$.
b) What will be the time of flight for this case?
c) What will be the time of flight to reach the highest point?
d) How high will be projectile go during this maximum-range trajectory?
e) What is the shape of the trajectory for projectile motion. I.e. what is $y = y(x)$?
f) If you wanted the projectile to land half as far away, to what would you set $\theta_0$?
g) Where does the weight of the projectile appear in your analysis? Explain.

Solution:

Observation:

$v_x = v_{x0} = v_0 \cos(\theta_0)$ is constant. Thus the distance travelled is $v_0 \cos(\theta_0) \cdot t$. So the initial angle that gives the greatest range is the angle that keeps the projectile up in the air the longest.

Since $a_y = g$,

$$y_f = \frac{1}{2} gt^2 + v_{y0}t + y_0$$

But $y_f = y_0 = 0$. So

$$0 = \frac{1}{2} gt + v_{y0} = \frac{1}{2} gt + v_0 \sin(\theta_0)$$

Thus

$$t = -\frac{2v_0 \sin(\theta_0)}{g}$$

This is the flight time whenever the starting point and ending point are at the same elevation in the gravitational field.

The range, $x_f$, is then

$$x_f = v_{0x} t + x_0 = v_{0x} t = v_0 \cos(\theta_0) t$$
\[ x_f = v_0 \cos(\theta_0) \left[ -\frac{2v_0 \sin(\theta_0)}{g} \right] = -\frac{2v_0^2 \cos(\theta_0) \sin(\theta_0)}{g} \]

A useful trigonometric identity here is

\[ \cos(\theta) \sin(\theta) = \frac{\sin(2\theta)}{2} \]

Thus

\[ x_f = -\frac{v_0^2 \sin(2\theta_0)}{g} \]

This applies whenever \( y_0 = y_f \). If the initial velocity is given, \( x_f \) is maximum when \( \sin(2\theta_0) = 1 \). \( \sin(90^\circ) = 1 \), so the maximum range occurs when \( \theta_0 = 45^\circ \). Thus

\[ x_f = -\frac{v_0^2}{g} \]

This is the answer to a) above. Note that \( g \) is negative, so the numerical value of the right-hand side is > 0.

When \( \theta_0 = 45^\circ \),

\[ t = -\frac{2v_0 \sin(\theta_0)}{g} = -\frac{\sqrt{2}v_0}{g} \]

This is the answer to b) above. The time of flight to reach the highest point is half the time to reach the impact point, since it is halfway along the trajectory in the \( x \)-direction. Thus

\[ t = -\frac{v_0}{\sqrt{2}g} \]

This is the answer to c) above.

At the maximum height, \( v_y = 0 \). Using the equation from above with this value for \( t \) and \( y_0 = 0 \),

\[ y_f = \frac{1}{2} g t^2 + v_{y0} t + y_0 \]

\[ y_f = \frac{1}{2} g \left( \frac{v_0}{\sqrt{2}g} \right)^2 - \frac{v_0 v_0}{2} \frac{v_0}{\sqrt{2}g} = \frac{v_0^2}{4g} - \frac{v_0^2}{2g} \]

\[ y_f = -\frac{v_0^2}{4g} \]

Or this could be gotten also from
\[ a_y \, dy = v_y \, dv_y \]
\[ g \int_{y_0}^{y_f} dy = \int_{v_{yo}}^{v_y} v_y \, dv_y \]
\[ g y_f = \int_{v_{yo}}^{0} v_y \, dv_y = -\frac{1}{2} v_{y0}^2 = -\frac{1}{2} \left( \frac{v_0}{\sqrt{2}} \right)^2 \]
\[ y_f = -\frac{v_0^2}{4g} \]

This is the answer to d) above.

To get \( y = y(x) \), we need to get our relationships for \( y = y(t) \) and \( x = x(t) \) and eliminate \( t \) from the relationship. Let’s do this for the general case, i.e. not for the maximum-range case.

\[ y = \frac{1}{2} g t^2 + v_{y0} t + y_0 \quad \text{and} \quad x = v_{0x} t + x_0 \]

With \( y_0 = 0 \) and \( x_0 = 0 \),

\[ y = \frac{1}{2} g t^2 + v_{y0} t \quad \text{and} \quad x = v_{0x} t \]

Thus

\[ t = \frac{x}{v_{0x}} \]

so

\[ y = \frac{1}{2} g \left( \frac{x}{v_{0x}} \right)^2 + v_{y0} \frac{x}{v_{0x}} \]

From this equation, it can be seen that the trajectory is a parabola. This answers e).

Regarding f), if the desired range is half the maximum range, what would \( \theta_0 \) be?

\[ x_f = -\frac{v_0^2 \sin(2\theta_0)}{g} = -\frac{v_0^2}{2g} \]

Thus

\[ \sin(2\theta_0) = \frac{1}{2} \]

Then

\[ \theta_0 = \frac{1}{2} \arcsin \left( \frac{1}{2} \right) = 15^\circ \]
But the gun can be raised up to 90°. In the angle range between 0° and 90°, \(2\theta_0\) goes from 0° to 180°. (See figure below.) Thus at \(\theta_0 = 75°\), \(\sin(150°) = 0.5\). This means that either 15° or 75° will halve the range, though each will have a different flight time.

Regarding g) and the weight of the projectile, it does not appear anywhere in this analysis. Does this mean that the weight doesn’t matter? No. The weight would enter into the picture by changing the initial velocity of the projectile. A heavier projectile will not be shot off as fast, given the amount of energy that can be conveyed to it at its launch.

This development applies to the flight of any kind of unpowered projectile, whether it be an artillery shell or a baseball. Let’s look at a historical case.

In 1918, the German army used a gigantic gun to bombard Paris. The gun was nicknamed “Big Bertha”. Big Bertha had a muzzle velocity of 1640 m/sec.
a) From what distance could the Germans bombard Paris?

Using $\theta_0 = 45^\circ$,

$$x_f = -\frac{v_0^2}{g} = -\frac{(1640 \text{ m/sec})^2}{-9.81 \text{ m/sec}^2} = 274200 \text{ m} = 274 \text{ m}$$

Wow! That’s about 170 miles! The Parisians thought that the Germans were bombing them from some secret, high-flying dirigible, because they never heard the shot from so far away.

b) How high would the projectiles for Big Bertha go in such a trajectory?

$$y_f = -\frac{v_0^2}{4g} = -\frac{(1640 \text{ m/sec})^2}{-4 \cdot 9.81 \text{ m/sec}^2} = 68540 \text{ m} = 68.5 \text{ m}$$

This also is quite a height, into the earth’s stratosphere. This was the highest that a man-made object went above the earth’s surface until the Germans started using ballistic rockets against England in WWII.

c) How long would the flight of a Big Bertha projectile last after it was fired until it landed in Paris?

$$t = -\frac{\sqrt{2}v_0}{g} = -\frac{\sqrt{2} \cdot 1640 \text{ m/sec}}{-9.81 \text{ m/sec}^2} = 236 \text{ sec}$$

This is almost 4 minutes!

Actually, the numbers for Big Bertha are somewhat different and smaller than these, I assume because of the effect of air resistance on such a fast-moving projectile.