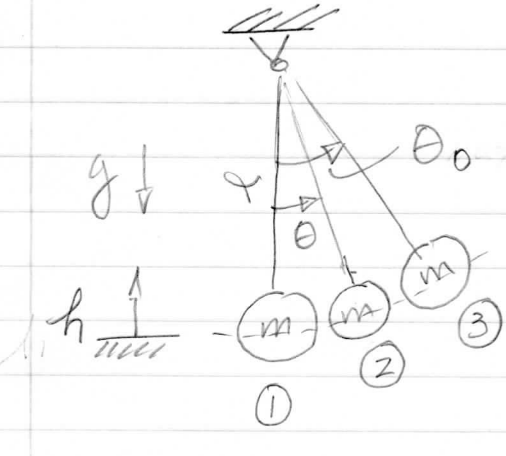
Pendulum equations via work/energy



At left is shown a pendelum in 3 different positions:

① At the bottom of its swing ($\theta = 0$)

② At an arbitrary position (θ)

③ At the top of its swing ($\theta = \theta_0$)

One sets the pendelum in motion by initially deflecting it to θ_0 & then releasing it from rest.

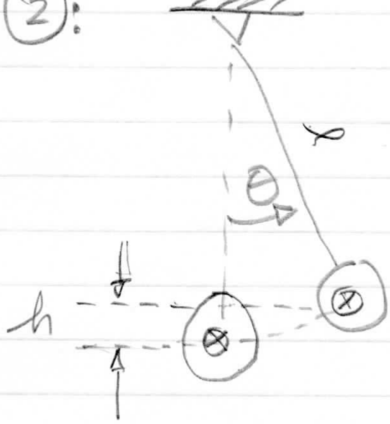
Since there are no non-conservative forces or moments in action (all friction is being neglected), the pendelum will swing forever.

The potential energy at the top of the swing will be converted to kinetic energy at the bottom of the swing, then converted back. In between, at θ, the energy is partially kinetic & partially gravitational potential (U_g).

So E, the total energy is constant:

$$E = U_{g_3} = T_1 = T_2 + U_{g_2}$$

(2):



$$h = l - l\cos\theta$$

So energy relationship between the 3 states is

$$E = \cancel{m}g(l - l\cos\theta_0) = \tfrac{1}{2}\cancel{m}v_1^2 \qquad (1)$$

$$= \tfrac{1}{2}\cancel{m}v^2 + \cancel{m}g(l - l\cos\theta)$$

Notice that the ms drop out.

From this we can calculate how fast the pendulum is going at $\theta = 0$; knowing its initial deflection, θ_0.

$$v_1 = \sqrt{2gl(1 - \cos\theta_0)} = v_{max}$$

This is the maximum velocity because at $\theta = 0$, all the energy is kinetic.

Let's look closer at the general state (2). There

$$E = \frac{1}{2} m v^2 + mg(l - l\cos\theta)$$

And $v = l\dot{\theta}$.

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$\frac{E}{ml} = \frac{1}{2} l \dot{\theta}^2 + g(1 - \cos\theta)$$

We can differentiate this w.r.t. t. The left side is constant, so

$$0 = \frac{1}{2} l \, 2\dot{\theta}\ddot{\theta} + g\sin\theta\,\dot{\theta}$$

$$0 = (l\ddot{\theta} + g\sin\theta)\,\dot{\theta}$$

This condition holds for all θ. At the top of the swing $\dot{\theta} = 0$, but elsewhere $\dot{\theta} \neq 0$. So we need the first factor $= 0$ to cover all other cases.

$$l\ddot{\theta} + g\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

We might also be interested in the tension in the cord, P. Since the pendulum is traveling a circular arc,

$$a_n = V^2 / l$$

From (1)

$$V^2 = 2gl(1-\cos\theta_0) - 2gl(1-\cos\theta)$$

$$V^2 = 2gl(\cos\theta - \cos\theta_0)$$

FBD = MAD

$$\uparrow + \Sigma F_n: \quad P - mg\cos\theta = m a_n = m\frac{V^2}{l}$$

$$P = m\left(\frac{V^2}{l} + g\cos\theta\right)$$

$$P = m\left[\frac{2g\cancel{l}(\cos\theta - \cos\theta_0)}{\cancel{l}} + g\cos\theta\right]$$

$$P = mg(3\cos\theta - 2\cos\theta_0)$$

Where is P max?

$$\frac{dP}{d\theta} = 0 = mg\,3(-\sin\theta)$$

At $\theta = 0$. There the velocity is highest and the weight acts directly against P.

$$P_{max} = mg\,(3 - 2\cos\theta_0)$$