Handy Analysis Sheet

Common notations and terminology

Sets will be defined by listing their elements or providing the criteria to be a member. If \( x \) is an element of the set \( A \), we write \( x \in A \); if not, we write \( x \notin A \). If every element of the set \( A \) is also an element of the set \( B \), then \( A \) is a subset of \( B \) and we write \( A \subseteq B \). If \( A \subseteq B \) and \( B \subseteq A \), then the sets are equal and we write \( A = B \).

NOTE: to prove two sets are equal, show that each is a subset of the other:

1. Suppose \( x \in A \). Then show that, based on this assumption, it follows that \( x \in B \). This shows \( A \subseteq B \).
2. Now suppose \( x \in B \). Then show that, based on this assumption, it follows that \( x \in A \). This shows \( B \subseteq A \).
3. Putting (1) and (2) together, this proves \( A = B \).

Special sets: each of these is is a subset of the next.

\( \emptyset = \{ \} \), the empty set; it contains no elements and is a subset of every other set.

\( N = \{1, 2, 3, \ldots \} \), the natural numbers

\( N_0 = \{0, 1, 2, 3, \ldots \} = \mathbb{N} \cup \{0\} \), also called the natural numbers

\( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \} \), the integers (“Zahlen” in German)

\( \mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\} \), the rational numbers (“quotients”)

\( \mathbb{R} = \{\ldots\} \), the real numbers

\( \mathbb{C} = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\} \), the complex numbers

Set operations:

intersection: \( A \cap B = \{x: x \in A \text{ and } x \in B\} \)
union: \( A \cup B = \{x: x \in A \text{ or } x \in B\} \)
complement: \( A^c = \{x: x \notin A\} \)
difference: \( A \setminus B = \{x: x \in A \text{ and } x \notin B\} = A \cap B^c \)
product: \( A \times B = \{(x, y): x \in A \text{ and } y \in B\} \)
containment: \( A \subseteq B \iff (x \in A \implies x \in B) \)

Subsets of \( \mathbb{R} \):

\((a, b) = \{x: a < x \text{ and } x < b\}\) is an open interval

\([a, b] = \{x: a \leq x \text{ and } x \leq b\}\) closed interval

\((a, b]\) or \([a, b)\) are half-open intervals

Logic and inference

“A implies B” is written \( A \implies B \) and means that if \( A \) is true, then \( B \) must also be true. This is if-then or implication. \( A \) is the hypothesis and \( B \) is the conclusion. To say “the hypothesis is satisfied” means that \( A \) is true. In this case, one can make the argument

\[
\begin{array}{c|c|c}
A & \implies B \\
\hline
A & B \\
\end{array}
\]

and infer that \( B \) must therefore be true, also. Logical equivalence: when \( A \implies B \) and \( B \implies A \), then the statements are equivalent and we write “\( A \) if and only if \( B \)” as \( A \iff B \), \( A \equiv B \), or \( A \text{ iff } B \).

Equivalent forms of an implication:

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<tr>
<th>( A )</th>
<th>( B )</th>
<th>( \neg(A \text{ and } \neg B) )</th>
<th>( \neg A \text{ or } B )</th>
<th>( \neg B \implies \neg A )</th>
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DeMorgan laws:

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<th>¬(A and B)</th>
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Set version of the DeMorgan laws:

\[(A \cap B)^c = A^c \cup B^c, \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c.\]

Distribution laws for sets:

\[A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad \text{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).\]

Containment law (set version of contrapositive):

\[A \subseteq B \iff B^c \subseteq A^c \iff A \cap B^c = \emptyset.\]

Universal quantifier: \(\forall x, A(x)\) means \(A(x)\) is true for all values of \(x\).

Existential quantifier: \(\exists x, A(x)\) means \(A(x)\) is true for some \(x\) (at least one, anyway).

Note: \(\exists! x, A(x)\) means there is a unique \(x\) for which \(A(x)\) is true.

Quantifier rules:

\[
\neg \exists x, A(x) \equiv \neg \forall x, A(x) \\
\exists x, \neg A(x) \equiv \forall x, \neg A(x) \\
\forall x, \forall y, A(x, y) \equiv \forall y, \forall x, A(x, y) \\
\exists x, \exists y, A(x, y) \equiv \exists y, \exists x, A(x, y) \\
\exists x, \forall y, A(x, y) \implies \forall y, \exists x, A(x, y)
\]

**Proof Techniques**

Often, you will need to prove a statement of the form \(A \implies B\).

Direct proof of \(A \implies B\):

(1) Assume the hypothesis \(A\), for the moment.
(2) Use this assumption, and whatever else you know, to prove that \(B\) is true.

Indirect proof: to prove \(A \implies B\), using the fact that \((A \implies B) \equiv (\neg B \implies \neg A)\).

(1) Assume, for the moment, that the opposite of the conclusion, \(\neg B\), is true.
(2) Use this assumption, and whatever else you know, to prove \(\neg A\) is true.

Contradiction: this works for proving statements that are not necessarily of the form \(A \implies B\). Suppose you are asked to show that some proposition \(P\) is true.

(1) Assume, for the moment, that \(P\) is false.
(2) Show that this assumption implies a fallacy (like \(x < x\), “9 is prime”, etc).

Mathematical induction: this works for proving statements which are supposed to be true for every natural number. To prove that \(P(n)\) is true whenever \(n \in \mathbb{N}\):

(1) Show \(P(1)\).
(2) Show that \(P(k) \implies P(k+1)\).

Note: \(\log x = \log_e x = \ln x\). If a different base is used, it will be specified, e.g.,

\[2^n < M \iff n < \log_2 M.\]