Series Practice, Math 413 Consider Prior to Midterm

These are just a few problems to get some series practice prior to our upcoming midterm. These may or may not appear later on actual HW 4, but they are mostly just a chance to apply what we’ve learned about series of functions. Let me know if you wish to discuss!

1. Consider each series of functions and determine a domain, if any, on which the series converge uniformly.
   \[\sum_{j=1}^{\infty} \frac{1}{x^2 + j^2}\]
   \[\sum_{j=1}^{\infty} \left(\frac{x + 1}{x - 1}\right)^j\]
   \[\sum_{j=1}^{\infty} \frac{1}{(x - j)^2}\]

2. Does the series
   \[\sum_{j=0}^{\infty} \frac{x^2}{(1 + x^2)^j}\]
   converge pointwise on \(\mathbb{R}\)? Does it converge uniformly?

3. Show that the function
   \[f(x) = \sum_{j=1}^{\infty} \frac{1}{j^2 + j^4x^2}\]
   converges uniformly on \(\mathbb{R}\)? At what values of \(x\) does \(f'(x)\) exist?

4. Show that the function
   \[f(x) = \sum_{j=1}^{\infty} \frac{\cos(jx)}{j^3}\]
   is integrable on any compact interval \([a, b]\).

5. Suppose \(f_j(x)\) is nonnegative and continuous and
   \[\sum f_j(x) = \frac{1}{\sqrt{5 - x^3}}\]
   for each \(x \in [0, 1]\). Show that the series \(\sum f_j\) converges uniformly on \([0, 1]\).