This homework set has eight (8) problems. Most of them are routine while some require more thought. You are encouraged to work with others and to ask questions of your instructor; however, you must write up your solutions independently. On this and all subsequent homework sets please write neatly and use complete sentences. Writing mathematics well is a craft, aim to hone it!

1. Prove the Bolzano-Weierstrass Theorem in $\mathbb{R}^n$.

2. Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

3. Formulate notions of boundedness for subsets $E \subset \mathbb{R}^n$ and for functions $f : \mathbb{R}^n \to \mathbb{R}$. Prove that the continuous image under $f : \mathbb{R}^n \to \mathbb{R}$ of a compact set is compact.

4. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x, y) = \begin{cases} 
   \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\
   0 & \text{if } (x, y) = (0, 0)
   \end{cases}$$

   has directional derivatives in all directions at the point $(0, 0)$, but is not continuous there.

5. Prove that $L : \mathbb{R}^n \to \mathbb{R}$ is linear if and only if there exists a vector $y \in \mathbb{R}^n$ such that for all $x$, $L(x) = \langle x, y \rangle$. Is this vector $y$ unique?

6. Prove that if $L : \mathbb{R}^n \to \mathbb{R}$ is linear, then $L$ is uniformly continuous on $\mathbb{R}^n$.

7. Let $f(x, y) = x^2 + y^2$ if $x$ and $y$ are both rational and $f(x, y) = 0$ otherwise. Show that $f$ is differentiable at $(0, 0)$. (Dirichlet’s revenge?)

8. Redo the proof from class that differentiability implies partial derivatives exist to show that differentiability implies all directional derivatives exist.