Homework # 5 Solutions, Math 413

1. Here we examine the function

\[
f(x) = \begin{cases} 
  e^{-1/x^2} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]

It is clear that we may differentiate \( f \) as many times as we like at any \( x \neq 0 \). To prove that \( f \) is infinitely differentiable at 0 as well, the following observations are helpful

1. \( f^{(k)}(x) = f(x) \cdot Q_k(x) \) where \( Q_k \) is some rational function (just a polynomial over another polynomial, we need not be any more specific). One can (and should) prove this by induction on \( k \).

2. \( \lim_{x \to 0} f(x)/p(x) = 0 \) for all polynomials \( p \). One can prove this by induction on the degree of \( p \) using L’Hospital’s Rule. In particular, \( \lim_{x \to 0} f(x) \cdot Q(x) = 0 \) for all rational functions \( Q \).

Now to prove that \( f \) is infinitely differentiable at 0, we show by induction that \( f^{(k)}(0) = 0 \) for all \( k \).

The case \( k = 1 \) is easily verified by L’Hospital’s Rule:

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = 0
\]

by observation (2).

Assuming \( f^{(k)}(0) = 0 \), we compute

\[
f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) \cdot Q_k(x)}{x} = 0
\]

by observations (1) and (2). Thus, \( f \) is infinitely differentiable at 0 and \( f^{(k)}(0) = 0 \) for all \( k \).

However, this forces the Taylor series centered at 0 to be identically zero since all the coefficients are zero. But our exponential function is never zero, so \( f \) is not identically zero on any neighborhood of 0. Thus, \( f \) and its own Taylor series disagree on every neighborhood of 0, i.e. \( f \) is not real analytic. \( \Box \)
2. Define \( h(x) = \cos^2(x) + \sin^2(x) \). By the series definitions of sine and cosine, it is easy to differentiate term by term and see that the derivative of sine is cosine and that the derivative of cosine is negative sine. Therefore

\[
h'(x) = 2 \cos(x)(-\sin(x)) + 2\sin(x)\cos(x) = 0
\]

So \( h \) must be constant. But from the series for sine and cosine it is evident that \( \sin(0) = 0 \) and \( \cos(0) = 1 \), so \( h(0) = 1 \). Therefore \( h(x) \equiv 1 \) as desired. □

3. We aim to show that \( e^x e^y = e^{x+y} \) for all real numbers \( x \) and \( y \). Fix \( y \) and define \( h(x) = e^{x+y} \). Note that \( h'(x) = h(x) \) by chain rule; we'll now show that only very certain functions can equal their own derivatives.

To wit, suppose \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f = f' \). Then \( f \) is \( C^\infty \) so we can at least form the Taylor series centered at \( c = 0 \):

\[
\sum_{j=0}^\infty \frac{f^{(j)}(0)}{j!} x^j. \tag{1}
\]

The question is whether or not this series actually represents \( f(x) \) at each \( x \in \mathbb{R} \). Examination of the remainder \( R_{N,0}(x) \) will tell the story:

\[
|R_{N,0}(x)| = \left| \int_0^x f^{(N+1)}(t) \frac{(x-t)^N}{N!} dt \right| \leq \int_0^x |f(t)| \frac{|x-t|^N}{N!} dt.
\]

But \( f \) is continuous on \([0,x]\), so \( |f(t)| \leq M \) on \( 0 \leq t \leq x \), thus continuing the inequalities we have

\[
|R_{N,0}(x)| \leq M \cdot \frac{|x|^{N+1}}{N!},
\]

which goes to 0 as \( N \to \infty \). So the series really does represent \( f(x) \) at each \( x \in \mathbb{R} \).

Now, since \( f^{(j)}(0) = f(0) \) for every \( j \), equation (1) reads

\[
\sum_{j=0}^\infty \frac{f(0)}{j!} x^j = f(0)e^x. \tag{2}
\]

So any function \( f : \mathbb{R} \to \mathbb{R} \) that satisfies \( f = f' \) is actually \( f(x) = f(0)e^x \). In particular, our function \( h(x) = e^{x+y} \) satisfies \( h(x) = h(0)e^x = e^y e^x \). □
4. Since $\pi/2$ is defined to be the first positive root of $\cos x$, we know that $\cos x > 0$ on $(0, \pi/2)$. Since $\cos x$ is the derivative of $\sin x$, we therefore know that $\sin x$ is strictly increasing on $(0, \pi/2)$, so it is one-to-one on this interval.

Moreover, since $\sin^2 x + \cos^2 x = 1$ for all $x$ and since $\cos^2(\pi/2) = 0$, we must have $\sin(\pi/2) = \pm 1$. But since $\sin 0 = 0$ and $\sin$ is increasing, we must have $\sin(\pi/2) = 1$. Since $\sin$ is continuous, the Intermediate Value Theorem guarantees that every value between 0 and 1 is attained. We have therefore shown that $\sin : (0, \pi/2) \to (0,1)$ is a bijection, hence is invertible.

Let $\arcsin : (0,1) \to (0, \pi/2)$ be the inverse of $\sin : (0, \pi/2) \to (0,1)$. Since the derivative of $\sin$ is $\cos$, which is never zero on $(0, \pi/2)$, $\arcsin$ is itself differentiable. To wit, let $b \in (0,1)$ with $\sin a = b$. Then

$$(\arcsin)'(b) = \frac{1}{(\sin)'(a)} = \frac{1}{\cos(a)} = \frac{1}{\sqrt{1 - \sin^2 a}} = \frac{1}{\sqrt{1 - b^2}}$$

Note that we used the positive square root above because we know $\cos$ is positive on $(0, \pi/2)$. □

5. The derivative of $\tan(x)$ is $\sec^2(x)$ by quotient rule. □

6. Let $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ be the inverse of $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. Since the derivative of $\tan$ is $\sec^2$, which is never zero on $(-\pi/2, \pi/2)$, $\arctan$ is itself differentiable. To wit, let $b \in (-\pi/2, \pi/2)$ with $\tan(a) = b$. Then

$$(\arctan)'(b) = \frac{1}{(\tan)'(a)} = \frac{1}{\sec^2(a)} = \frac{\cos^2(a)}{1 + b^2} \quad \square$$

7. The largest neighborhood of $c = 1$ on which $f(x) = \ln(x)$ is real analytic is $(0,2)$. This looks right from the graph $y = \ln(x)$, but was hard (to me, anyway) to show directly from the remainder. The good news is we don’t have to!

Observe that

$$\ln(1 + x) = \int \frac{1}{1 + x} \, dx = \int \left( \sum_{j=0}^{\infty} (-1)^i x^i \right) \, dx,$$

provided $|x| < 1$. Since there is only one series (the Taylor series) that a function can be and we are free to integrate power series term-by-term, we
get
\[ \ln(1 + x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}. \]

If we directly build the Taylor series for \( f(x) = \ln(x) \) centered at \( c = 1 \), however, we get
\[
\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (x - 1)^j = \ln(1 + (x - 1)) = \ln(x),
\]
provided \( |x - 1| < 1 \). So \( \ln(x) \) equals its own Taylor series on \((0, 2)\). \( \square \)