Homework # 4 Solutions, Math 413

1. The hypotheses on this one are exactly those of Dini’s theorem, which you proved on HW #3, applied to the sequence \{S_N\} of partial sums of the given series. Thus the convergence is uniform. □

2. The series

\[ f(x) = \sum_{j=1}^{\infty} \frac{1}{j^2 + j^4x^2} \]

converges uniformly on \( \mathbb{R} \) by M-test with \( M_j = 1/j^2 \). Now, the derived series

\[ f(x) = \sum_{j=1}^{\infty} \frac{-2j^4x}{(j^2 + j^4x^2)^2} \]

is a bit more involved. The natural bounds to try M-test here are \( M_j = \frac{2}{j^2x^2} \) but these are not helpful unless \( x \) is bounded away from the origin. So on any set of the form \((-\infty, a] \cup [a, \infty)\) the derived series converges uniformly and therefore the original series is differentiable. Thus, \( f'(x) \) exists at any \( x \neq 0 \).

As for \( x = 0 \), the jury is currently out but folks are now leaning toward not differentiable there. I’m still working on it... □

3. Let \( \sum_{j=0}^{\infty} a_j(x-c)^j \) be a convergent power series with radius of convergence \( \rho \), and let \( \sum_{j=1}^{\infty} ja_j(x-c)^{j-1} \) be its derived series. We “proved” in class that the derived series also has radius of convergence \( \rho \). Here we rigorously defend the moves we made in class.

We first consider the case \( 0 < \rho < \infty \). In this case, \( \rho = 1/A \) where

\[ A = \lim_{j \to \infty} \sup |a_j|^{1/j} \]

Reindexing the derived series as \( \sum_{j=0}^{\infty} (j+1)a_{j+1}(x-c)^j \), we hope to show that the Hadamard formula applied to the derived series yields the same radius of convergence. In other words, we aim to prove that if

\[ B = \lim_{j \to \infty} \sup |(j+1)a_{j+1}|^{1/j}, \]

then \( B = A \). By definition, \( A = \lim A_j \) and \( B = \lim B_j \) where

\[ A_j = \sup\{|a_k|^{1/k} : k \geq j\} \]

and \( B_j = \sup\{|(k+1)a_{k+1}|^{1/k} : k \geq j\} \)

Here is one potential execution of our outline developed last week:
(I) Show that $B \leq A$. Seeking a contradiction, suppose $B > A$. For $\varepsilon = (B - A)/3$, there exists $N \in \mathbb{N}$ such that for $j \geq N$ we have $|a_{j+1}|^{1/(j+1)} < A + \varepsilon$. Multiplying both sides by $(j + 1)^{1/j}$, we obtain

$$(j + 1)^{1/j}|a_{j+1}|^{1/(j+1)} < (j + 1)^{1/j}(A + \varepsilon).$$

But by exponent arithmetic, we can rewrite to obtain

$$|(j + 1)a_{j+1}|^{1/j}|a_{j+1}|^{-1/j(j+1)} < (j + 1)^{1/j}(A + \varepsilon).$$

Clearing the negative powers, we finally obtain

$$|(j + 1)a_{j+1}|^{1/j}|a_{j+1}|^{-1/j(j+1)} < (j + 1)^{1/j}(A + \varepsilon) |a_{j+1}|^{1/j(j+1)} < (j + 1)^{1/j}(A + \varepsilon)^{(j+1)/j}.$$  

When $j$ is large we have a contradiction because the right side tends to $A + \varepsilon < B$ (yet $B$ is the decreasing limit of the $B_j$).

(II) Show that $B \geq A$.

(a) Claim 1: For each $\varepsilon > 0$, 

$$A - \varepsilon < |a_j|^{1/j}$$

for infinitely many values of $j$.

Pf: This is by the definition of lim sup; if $A - \varepsilon < |a_j|^{1/j}$ holds for only finitely many $j$, then we would have $A_j < A$ for large $j$.

(b) Claim 2: For each $\varepsilon > 0$, 

$$A - \varepsilon < |(j + 1)a_{j+1}|^{1/j}$$

for infinitely many values of $j$.

Pf: By Claim 1, there are infinitely many $j$ for which 

$$A - \varepsilon < |a_{j+1}|^{1/(j+1)}.$$ 

For these values of $j$,

$$(j + 1)(A - \varepsilon) < (j + 1)|a_{j+1}|^{1/(j+1)}.$$
Of course, only finitely many $j$ satisfy $(j + 1)(A - \varepsilon) < 1$, so for infinitely many $j$ we must have

$$1 < (j + 1)|a_{j+1}|^{1/(j+1)}.$$ 

Taking $j$th roots, we finally obtain

$$1 < (j + 1)^{1/j}|a_{j+1}|^{1/(j+1)}.$$ 

The rest is just algebra now, subtracting 1 from both sides and multiplying by $|a_{j+1}|^{1/(j+1)}$ we get

$$0 < |a_{j+1}|^{1/(j+1)} \cdot [(j+1)^{1/j}|a_{j+1}|^{1/(j+1)} - 1] = (j+1)^{1/j}|a_{j+1}|^{1/j} - |a_{j+1}|^{1/(j+1)}.$$ 

So at long last,

$$(j + 1)^{1/j}|a_{j+1}|^{1/j} > |a_{j+1}|^{1/(j+1)} > A - \varepsilon$$

for infinitely many $j$.

(c) Claim 3: For each $\varepsilon > 0$, $B \geq A - \varepsilon$.

Pf: This is now a straightforward lim sup argument applying Claim 2 to the definition of $B$.

We have thus shown both $B \leq A$ and $A \leq B$, so $A = B$ and the proof is complete! □

Remark: For the cases $A = 0, \infty$ a similar (simpler?) limiting argument works.
4. Let \( f \) be infinitely differentiable on an interval \( I \). Suppose \( a \in I \) and there exist constants \( C, R \) such that for all \( x \) in some neighborhood of \( a \) it holds that

\[
|f^{(k)}(x)| \leq C \cdot \frac{k!}{R^k}
\]

We must show that under these conditions there is a neighborhood of \( a \) on which \( f \) agrees with its Taylor series centered at \( a \).

Let \( J \subseteq I \) be the neighborhood of \( a \) on which the inequality holds. Choose \( \delta < R \) so small that \( (a - \delta, a + \delta) \subseteq J \).

We estimate the remainder from Taylor’s theorem on this \( \delta \)-neighborhood of \( a \):

Let \( x \in (a - \delta, a + \delta) \). Then we have

\[
|R_{k,a}(x)| = \left| \int_{a}^{x} f^{(k+1)}(t) \frac{(x-t)^k}{k!} \, dt \right|
\]

\[
\leq \int_{a}^{x} |f^{(k+1)}(t)| \frac{|x-t|^k}{k!} \, dt
\]

\[
\leq \int_{a}^{x} C \cdot \frac{(k+1)!}{R^{k+1}} \frac{|x-t|^k}{k!} \, dt
\]

\[
= C \cdot \frac{(k+1)!}{R^{k+1}} \int_{a}^{x} \frac{|x-t|^k}{k!} \, dt
\]

\[
\leq C \cdot \frac{(k+1)!}{R^{k+1}} \int_{a}^{x} \frac{|x-a|^k}{k!} \, dt
\]

\[
= C \cdot \frac{(k+1)!}{R^{k+1}} |x-a|^k \int_{a}^{x} 1 \, dt
\]

\[
= C \cdot \frac{(k+1)!}{R^{k+1}} |x-a|^k \int_{a}^{x} 1 \, dt
\]

\[
\leq C \cdot \frac{(k+1)!}{R^{k+1}} \frac{\delta^{k+1}}{R^{k+1}}
\]

We need to show that the remainder \( R_{k,a}(x) \to 0 \), so it suffices to show that \( (k+1)\delta^{k+1}/R^{k+1} \to 0 \). But this is not difficult because the series

\[
\sum (k+1)(\delta/R)^{k+1}
\]

converges by the Ratio Test (since \( \delta/R < 1 \)). Thus the terms go to zero as desired. \( \square \)
5. This problem says “If the zeros of a real analytic function have an accumulation point, then the function is identically zero”. This is a handy thing to know.

Now on to the problem at hand. Let $a$ be an accumulation point for the set

$$Z = \{ x \in (-r, r) : f(x) = 0 \}$$

consisting of the zeros of $f$.

By problem six below, $f$ may expanded in a Taylor series centered at $a$ that is valid on some (possibly small) neighborhood of $a$ contained within the original interval $(-r, r)$. For the moment, call this neighborhood $J$.

Since $a$ is an accumulation point for $Z$, there is a sequence of points in $Z$ (i.e. zeros of $f$) that converges to $a$. But $f$ is continuous on $(-r, r)$, so this forces $f(a) = 0$.

What’s more, $f'(a) = 0$. To wit, let $x_n \in Z$ converge to $a$.

$$f'(a) = \lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n \to \infty} 0 = 0$$

It is helpful to note that $a$ is also an accumulation point for the zeros of $f'$. Indeed, Rolle’s theorem guarantees that between each of the $x_n$ in $Z$ there is a point $y_n$ such that $f'(y_n) = 0$. Applying the whole line of reasoning with $f$ replaced by $f'$ and $f'$ replaced by $f''$, we see that $f''(a) = 0$ and that $a$ is an accumulation point for the zeros of $f''$. Analogously, one could use induction to show that $f^{(k)}(a) = 0$ for all $k$.

So the expansion

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x - a)^j$$

is identically zero, which implies that $f$ itself is identically zero on $J$ since they agree on this interval. But now the endpoints of $J$ are accumulation points for $Z$, so we could do the same routine all over again at these endpoints.

This implies that the supremum and infimum of the set of accumulation points of $Z$ are in fact $r$ and $-r$, respectively. Thus, $f$ is identically zero on the entire original interval. □
6. Let \( f(x) = \sum_{j=0}^{\infty} a_j (x - c)^j \) with interval of convergence \((c - \rho, c + \rho)\). We claim that this function \( f \) is real analytic on the set \((c - \rho, c + \rho)\). That is, given \( d \in (c - \rho, c + \rho) \) we must show that the Taylor series of \( f \) about \( d \) converges to \( f \) on a neighborhood of \( d \).

Note that \( f \in C^\infty((c - \rho, c + \rho)) \) so that we may at least form the Taylor series at any point we like. This follows exactly because term by term differentiation of \( f \) is valid anywhere inside \((c - \rho, c + \rho)\).

Now, according to problem 4 above, it suffices to show that for each \( d \in (c - \rho, c + \rho) \setminus \{c\} \) there exists a neighborhood \( J \) and constants \( C \) and \( R \) such that

\[
|f^{(k)}(x)| \leq C \frac{k!}{R^k}
\]

for every \( x \) in \( J \) and all \( k \).

Fix \( d \in (c - \rho, c + \rho) \setminus \{c\} \). Let \( J = (d - \delta, d + \delta) \) be such that \( 0 < \delta < |c - d| \) and \( 0 < \delta < \rho - |c - d| \). Draw a picture of this, we’re just choosing \( \delta \) so that \( J \) does not contain the original center \( c \) and also does not share any boundary points with \((c - \rho, c + \rho)\). As such, we may now choose \( L \) satisfying \(|c - d| + \delta < L < \rho\).

Because \( \rho^{-1} = \limsup_{j \to \infty} |a_j|^{1/j} \), there exists \( M > 0 \) such that

\[
|a_j| \leq \frac{M}{L^j} \quad \forall j.
\]  

This is the corollary to the Hadamard formula in your text, which comes exactly from the Hadamard formula and definition of \( \lim sup \) (Do not just tune out here, this is a straightforward and short 412 argument with \( \lim sup \)!).

We are now ready to estimate \(|f^{(k)}(x)|\) for any \( x \in J \):

\[
|f^{(k)}(x)| = \left| \sum_{j=k}^{\infty} \frac{k!}{j!} a_j (x - c)^{j-k} \right|
\]

\[
\leq k! \sum_{j=k}^{\infty} \binom{j}{k} |a_j| \cdot |x - c|^{j-k}
\]

\[
\leq \frac{k!M}{|x - c|^k} \sum_{j=k}^{\infty} \binom{j}{k} \left( \frac{|x - c|}{L} \right)^j \quad \text{by (1)}
\]

The remarkable and lovely thing about the series in the last line of the
display above is that we can sum it! To wit,
\[ \sum_{j=0}^{\infty} \left( \frac{j}{k} \right) z^j = \frac{z^k}{k!} \frac{d^k}{dz^k} \left( \sum_{i=0}^{\infty} z^i \right) = \frac{z^k}{k!} \frac{d^k}{dz^k} \left( \frac{1}{1-z} \right) = \frac{z^k}{(1-z)^{k+1}}, \]
valid whenever \(|z| < 1\). Since \(|x - c| < L\), we therefore have
\[ \sum_{j=k}^{\infty} \left( \frac{j}{k} \right) \left( \frac{|x-c|}{L} \right)^j = \frac{\left( \frac{|x-c|}{L} \right)^k}{(1 - \frac{|x-c|}{L})^{k+1}} = \frac{L|x-c|^k}{(L - |x-c|)^{k+1}} \]
Thus, continuing from the bottom of page 5, we find that
\[ |f^{(k)}(x)| \leq \frac{k!M}{|x-c|^k} \frac{L|x-c|^k}{(L - |x-c|)^{k+1}} \leq \frac{ML}{L - (|c-d| + \delta)} \frac{k!}{(L - (|c-d| + \delta))^k} \equiv C \frac{k!}{R^k}. \quad \square \]

**Remarks:** This is not easy. Let us consider ourselves brave/privileged to have carefully justified this statement and walk with our heads high whenever we assert that a power series is real analytic on its own interval of convergence.

Also, some of you used a recenter and swap double summation approach. This is excellent and requires its own brand of justification. See the advanced calculus books of Apostol and Rudin for further thoughts on this approach.

7. We are asked to show that any solution of the differential equation \( f'(x) + f(x) = x \) is necessarily real analytic. This is a lot like our lemma from class where we proved that solutions of \( f'' = -f \) have a certain form. Along the way in that proof we showed that solutions had to be real analytic by estimating the remainder \( R_{k,a}(x) \) and showing it goes to zero as \( k \to \infty \).
This we do here too:

\[ |R_{k,a}(x)| = \left| \int_a^x f^{(k+1)}(t) \frac{(x-t)^k}{k!} \, dt \right| \]
\[ \leq \int_a^x |f^{(k+1)}(t)| \frac{|x-t|^k}{k!} \, dt \]
\[ \leq \int_a^x M \frac{|x-t|^k}{k!} \, dt \]
\[ \leq \frac{M}{k!} |x - a|^{k+1} \]

The second “≤” comes from the fact that the condition \( f'(x) + f(x) = x \) implies that any derivative is some amalgamation of \( f, f' \) and the constant \( 1 \), hence bounded by some \( M \) on \([a, x]\) for each \( x \). Now the last term

\[ \frac{M}{k!} |x - a|^{k+1} \to 0 \]

as \( k \to \infty \) because the series \( \sum \frac{z^{k+1}}{k!} \) converges by Ratio Test. □