1. Find the pointwise limit for each of the sequences of functions defined below, then decide if the convergence is uniform.

(a) \( f_j(x) = jx(1 - x)^j \quad x \in [0, 1] \)

It is clear on this one that if \( x = 0, 1 \), then \( f_j(x) = 0 \) for all \( j \), so the pointwise limit at these endpoints is zero. In case \( 0 < x < 1 \), we must evaluate \( \lim_{j \to \infty} jx(1 - x)^j \). This limit is of the indeterminate form \( \infty \cdot 0 \), so we might try L'Hospital's Rule. Alternatively, we could observe that the infinite sum \( \sum_j ja^j \) converges when \( |a| < 1 \) (by the Ratio Test, for instance), so the terms \( ja^j \) themselves must go to zero. Setting \( a = 1 - x \), we get \( \lim_{j \to \infty} jx(1 - x)^j \leq \lim_{j \to \infty} ja^j \). Thus, \( \lim_{j \to \infty} jx(1 - x)^j = 0 \) and the pointwise limit on \([0, 1]\) is the zero function.

This convergence is not uniform, however. By calculus, we find that the absolute maximum of \( f_j \) on \([0, 1]\) occurs at \( x = 1/(j + 1) \) and that \( f_j(1/(j + 1)) = (j/(j + 1))^j \). This maximum value does not approach zero as \( j \) gets large (what does it approach?), so the convergence to the zero function is not uniform.

(b) \( f_j(x) = x^{2j}/(1 + x^{2j}) \quad x \in \mathbb{R} \)

We observe that when \( |x| < 1 \), \( f_j(x) \to 0 \); when \( |x| = 1 \), \( f_j(x) \to 1/2 \); and when \( |x| > 1 \), \( f_j(x) \to 1 \). So the pointwise limit is discontinuous at \( \pm1 \) and , since each individual \( f_j \) is continuous everywhere, the convergence cannot be uniform.

(c) \( f_j(x) = \sin jx/j \sqrt{x} \quad x \in (0, \infty) \)

There are two estimates that will be of use in our analysis of this sequence:

1. \( |\sin jx/j \sqrt{x}| \leq 1/j \sqrt{x} \) for all \( x \in (0, \infty) \);
2. \( |\sin jx| \leq jx \) for all \( x \in (0, \infty) \).

Estimate 1 follows since \( |\sin jx| \leq 1 \) for any \( x \). This one proves that the pointwise limit is zero at every fixed \( x \). We will also use it later to help prove uniform convergence.

Estimate 2 follows by a geometric argument with the unit circle, or alternatively, if you’re willing to accept certain things about the sine function, by calculus. Draw the sine curve on the positive \( x \)-axis and the tangent line \( y = x \) at the origin. The fact that the curve lies between -1 and 1 and is
always below the tangent line proves that $|\sin x| \leq x$ for all positive $x$. But if $x$ is positive, so is $\dot{x}$ (and vice versa).

Now on to the proof that the convergence to the zero function is uniform. Let $\varepsilon > 0$. We must prove that for $j$ big enough, $|f_j(x)| < \varepsilon$ for all $x > 0$. We will dispatch all possible $x$ values in two groups.

First, if $x \geq \varepsilon^2$, let $N > 1/\varepsilon^2$. Then for $j \geq N$ estimate 1 yields

$$\left| \frac{\sin jx}{j\sqrt{x}} \right| \leq \frac{1}{j\sqrt{x}} < \frac{1}{N\varepsilon^2} < \varepsilon$$

Second, if $0 < x < \varepsilon^2$, then rewrite estimate 2 as $|\sin jx/j\sqrt{x}| \leq \sqrt{x}$. In this case, no matter the value of $j$, we get $|\sin jx/j\sqrt{x}| < \sqrt{\varepsilon^2} = \varepsilon$.

Therefore, if $N > 1/\varepsilon^2$ then $j \geq N$ implies $|f_j(x)| < \varepsilon$ for all $x > 0$ and the convergence is uniform. $\square$

2. Suppose our sequence $f_j \rightarrow f$ uniformly on $[0, 1]$. Then $f$ is continuous (since each $f_j$ is continuous by hypothesis), hence $f$ is bounded on the compact set $[0, 1]$. Say $|f(x)| \leq K$ for all $x \in [0, 1]$. Since $f_j \rightarrow f$ uniformly, there exists $N$ such that if $j \geq N$, then $|f_j(x) - f(x)| < 1$. Therefore if $j \geq N$, $|f_j(x)| \leq K + 1$ for all $x$.

Now, each of $f_1, f_2, f_3, \ldots, f_{N-1}$ is itself continuous, hence bounded, on $[0, 1]$. For each $j = 1, \ldots, N - 1$ let $|f_j(x)| \leq M_j$ for all $x \in [0, 1]$. Setting $M = \max\{K + 1, M_1, \ldots, M_{N-1}\}$, we have $|f_j(x)| \leq M$ for all $j$ and all $x \in [0, 1]$. $\square$

If we reduce the convergence to pointwise, the result is no longer valid. Consider the functions $f_j$ given by

$$f_j(x) = \begin{cases} 2j^2x & \text{if } 0 \leq x \leq 1/2j \\ -2j^2x + 2j & \text{if } 1/2j \leq x \leq 1/j \\ 0 & \text{if } 1/j \leq x \leq 1 \end{cases}$$

This may look like it came out of left field, but each $f_j$ is just a triangle of height $j$ and base $1/j$ that has its bottom left corner at the origin, and then identically zero on the rest of the interval $[0, 1]$. Since $1/j \rightarrow 0$, these $f_j$ approach 0 pointwise. But they get taller and taller as $j$ grows, so there is no single $M$ that can bound them all.

3. Again, since each $f_j$ is continuous and they converge to $f$ uniformly, $f$
itself is continuous. We are asked to show that at each particular \( x \in \mathbb{R} \)
\[
\lim_{j \to \infty} f_j(x + 1/j) = f(x)
\]
So fix a particular \( x \) and let \( \varepsilon > 0 \). Since the convergence is uniform, there exists \( N_1 \) such that \( j \geq N_1 \) implies \( |f_j(y) - f(y)| < \varepsilon/2 \) for all \( y \). Now, since \( f \) is continuous at \( x \), there exists \( \delta > 0 \) such that \( |y - x| < \delta \) implies \( |f(y) - f(x)| < \varepsilon/2 \). Choose \( N_2 \) such that \( 1/N_2 < \delta \) and set \( N = \max\{N_1, N_2\} \).

For \( j \geq N \) we then have
\[
|f_j(x+1/j) - f(x)| \leq |f_j(x+1/j) - f(x+1/j)| + |f(x+1/j) - f(x)| < \varepsilon/2 + \varepsilon/2
\]
which completes the proof. \( \square \)

4. On this one we already have each \( f_j \) defined and continuous on all of \([a, b]\), but the limit \( f \) is initially defined only on \((a, b)\). We wish to somehow define \( f \) at the endpoints so that the convergence becomes uniform on the whole closed interval \([a, b]\).

If this is to work out, we at least have to define \( f(a) \) and \( f(b) \) so that \( f \) is continuous at these endpoints (otherwise there is no hope that \( f \) could be the uniform limit of functions that \textit{are} continuous there). So we need
\[
\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b} f(x) = f(b)
\]
Since each \( f_j \) is continuous, \( \lim_{x \to a} f_j(x) = f_j(a) \). Thus, our limit swapping theorem guarantees that
\[
\lim_{x \to a} f(x) = \lim_{j \to \infty} \lim_{x \to a} f_j(x),
\]
provided \( \lim_{j \to \infty} f_j(a) \) exists at all. (To see that this limit exists, show that \( \{f_j(a)\} \) is Cauchy.)

So we just \textit{define} \( f(a) \) to be this limit: \( f(a) \equiv \lim_{x \to a} f(x) \). Define \( f(b) \) similarly.

That these definitions result in uniform convergence on the whole interval \([a, b]\) is almost an afterthought now. To wit, we still have the uniform convergence on \((a, b)\) that we began with. Throwing in two more points at the endpoints can’t affect this convergence. After all, even if you have to go out a little further on the list of functions to get close to \( f \) at these endpoints, there are only two of them, so you’ll eventually be far enough on the list to be close to \( f \) at every point. Provide the details of these assertions. \( \square \)
5. The critical points in the proof of Dini’s Theorem are that the $f_j$ are increasing (decreasing would also do) and that the underlying domain is compact. For me it is easiest to prove this result in the contrapositive. We will show that if $f_j$ does not converge to $f$ uniformly, then it does not even converge pointwise.

Assume $f_j$ does not converge to $f$ uniformly on $K$. Then there is some $\varepsilon > 0$ such that no matter how big $N$ is, we can find $j \geq N$ such that it is not true that $|f_j(x) - f(x)| < \varepsilon$, i.e. there exists at least one $x$ with $|f_j(x) - f(x)| \geq \varepsilon$.

Since the $f_j$ are increasing, if any one of them, say $f_{j_0}$, satisfies $|f_{j_0}(x) - f(x)| < \varepsilon$ for all $x$, then every $f_j$ with $j \geq j_0$ satisfies $|f_j(x) - f(x)| < \varepsilon$ for all $x$. Since this cannot happen when the convergence is not uniform (see the previous paragraph), to each $f_j$ there corresponds at least one point $x_j \in K$ such that $|f_j(x_j) - f(x_j)| \geq \varepsilon$. Form a sequence $\{x_j\}$ from these points.

Since $K$ is compact, this sequence has an accumulation point $x$ in $K$. The claim is that, for this particular $x$, $\lim_{j \to \infty} f_j(x) \neq f(x)$. This would prove that the $f_j$ don’t even converge pointwise on $K$ and would therefore complete our proof.

Showing $\lim_{j \to \infty} f_j(x) \neq f(x)$ is not difficult. We know already that $|f_1(x_1) - f(x_1)| \geq \varepsilon$. But more is true as a consequence of the fact that at each point the $f_j$ are increasing. So $|f_2(x_2) - f(x_2)| \geq \varepsilon$ implies $|f_1(x_1) - f(x_2)| \geq \varepsilon$. In fact, the same reasoning shows that $|f_1(x_j) - f(x_j)| \geq \varepsilon$ for all $j \geq 1$. Since $f_1$ and $f$ are both continuous and since $x$ is an accumulation point of $\{x_j\}$, this finally implies $|f_1(x) - f(x)| \geq \varepsilon$.

But the reasoning of the last paragraph may be applied to $f_2, f_3, \ldots$ as well. So $|f_j(x) - f(x)| \geq \varepsilon$ for every $j$. Thus, $f_j$ does not converge to $f$ at the point $x$. $\square$

**Remark:** Many of you gave lovely direct proofs of this using Heine-Borel. You decide which you prefer! See problem 6 for a similar flavor.

6. The answer is NO. Consider the sequence $f_j$ defined on $\mathbb{R}$ by $f_j(x) = x - 1/j$. The graphs of these are a bunch of straight lines that are the line $y = x$ shifted down by $1/j$. Since $1/j$ tends to 0, this sequence clearly converges to $f(x) = x$ uniformly. However,

$$|f_j^2(x) - f^2(x)| = |(x^2 - 2x/j + 1/j^2) - x^2| = | - 2x/j + 1/j^2|$$

So, for example, when $x = j$ we have $|f_j^2(x) - f^2(x)| = 2 - 1/j^2 \geq 1$. Thus,
no matter how big $j$ gets we can find points $(x = j)$ where $|f_j^2(x) - f^2(x)|$ is not smaller than 1. Thus the convergence is not uniform.

If you add the simple hypothesis that $f$ be bounded, then $f_j \to f$ uniformly does imply $f_j^2 \to f^2$ uniformly. For if $f$ is bounded and $f_j \to f$ uniformly, then eventually $f_j$ is uniformly close to $f$, hence bounded as well. Thus, for $j$ big enough, the quantity $|f_j(x) + f(x)| \leq M$ for some $M$.

Now, since $|f_j^2(x) - f^2(x)| = |f_j(x) - f(x)| \cdot |f_j(x) + f(x)|$, big $j$ will make the second term on the right less than or equal to $M$, and big $j$ can make the first term less than $\varepsilon/M$. Consequently, $|f_j^2(x) - f^2(x)|$ can be made as small as we like provided that $f$ is bounded.

7. This problem illustrates that Lipschitz functions are indeed quite special. Observe first that each $f_j$ is uniformly continuous (proved easily from the Lipschitz condition) and, more importantly, the given pointwise convergence is enough to ensure that $f$ itself inherits the Lipschitz condition.

We have to show that $f_j \to f$ uniformly, so let $\varepsilon > 0$. Since $|f(s) - f(t)| < |s - t|$, for $\delta = \varepsilon/3$ we have that $|s - t| < \delta$ implies $|f(s) - f(t)| < \varepsilon/3$. Cover $[0,1]$ extremely inefficiently with open $\delta$-neighborhoods around every point:

$$[0,1] \subseteq \bigcup_{x \in [0,1]} (x - \delta, x + \delta)$$

Since $[0,1]$ is compact, there exists a finite subcover (Heine-Borel Theorem). That is, there exist finitely many points $\{x_1, x_2, \ldots, x_n\}$ such that

$$[0,1] \subseteq \bigcup_{k=1}^n (x_k - \delta, x_k + \delta)$$

In other words, every point in $[0,1]$ is within $\delta$ of one of the points $x_k$.

We are given that $f_j \to f$ pointwise, so for each $k = 1, \ldots, n$ there exists $N_k$ such that $j \geq N_k$ implies $|f_j(x_k) - f(x_k)| < \varepsilon/3$. Set $N = \max\{N_1, \ldots, N_n\}$.

Then for any $x$ in $[0,1]$ we have $|x - x_k| < \delta$ for one of the $x_k$. Thus, $j \geq N$ implies

$$|f_j(x) - f(x)| \leq |f_j(x) - f_j(x_k)| + |f_j(x_k) - f(x_k)| + |f(x_k) - f(x)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$$< \varepsilon$$

$\square$