1) Suppose $f$ is Riemann integrable on $[a,b]$, say with

\[ A = \int_a^b f(x) \, dx \]

Then by definition there exists $\delta > 0$ such that for any two partitions $P, P'$ with mesh $m(P), m(P') < \delta$ we have $|R(f, P) - R(f, P')| < 1$. Let $P = P'$ be any fixed partition (of distinct points) of mesh less than $\delta$ and pick sample points $s_j$ for $P$ however you like. Set $\Delta = \min \Delta_j$ and $M = \max |f(s_j)|$. We claim that $|f(x)| \leq M + 1/\Delta$ for all $x$ in $[a, b]$.

Let $x \in [a, b]$. Then $x$ is in one of the subintervals $I_j$ of the partition $P$, say $x \in I_k$. We choose sample points $s'_j$ for the partition $P'$ as follows: for $j \neq k$, let $s'_j = s_j$, but in the $k$th subinterval let $s'_k = x$. Then we have

\[ 1 > |R(f, P) - R(f, P')| = |f(s_k) - f(x)| \cdot \Delta_k \geq |f(s_k) - f(x)| \cdot \Delta \]

Therefore, $|f(s_k) - f(x)| < 1/\Delta$. Since $|f(s_k)| \leq M$, we finally get

\[ |f(x)| \leq M + 1/\Delta \]

as desired. \( \Box \)

2) Most of you mimicked the proof of (a) from class just fine. Let me know if you wish to discuss either part of that proposition.

3) The Dirichlet function (named after yet another of your mathematical ancestors) is not Riemann integrable. This can be seen by noting that no matter how small a mesh you take for a partition, there will always be both rational and irrational numbers in every subinterval $I_j$. By taking your sample points $s_j$ to be rational, every Riemann sum totals $(b - a)$; by taking your sample points $s_j$ to be irrational, every Riemann sum totals 0. Since this cannot be remedied by taking a finer partition, the function is not Riemann integrable. \( \Box \)

4) You guys have already met this function and shown it to be continuous everywhere. All continuous functions on closed intervals are integrable. (One can show continuity at the origin by observing that $0 \leq |x \sin(1/x)| \leq |x|$ and employing squeeze theorem; away from the origin $f$ is an algebraic combination of continuous functions.) \( \Box \)
5) We are asked to prove:

**The Upper and Lower Sum Characterization:** The function \( f \) is Riemann integrable if and only if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( \mathcal{P} = \{a, x_1, x_2, \ldots, x_{k-1}, b\} \) is a partition of \([a, b]\) with \( m(\mathcal{P}) < \delta \) we have

\[
\sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j < \varepsilon
\]

where \( I_j = [x_{j-1} - x_j] \).

\((\Rightarrow)\) Suppose \( f \) is Riemann integrable and let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that

\[
m(\mathcal{P}), m(\mathcal{P}') < \delta \Rightarrow |\mathcal{R}(f, \mathcal{P}) - \mathcal{R}(f, \mathcal{P}')| < \frac{\varepsilon}{3}.
\]

Now, let \( \mathcal{P} = \{a, x_1, x_2, \ldots, x_{k-1}, b\} \) be an arbitrary partition of \([a, b]\) with \( m(\mathcal{P}) < \delta \). Choose sample points \( \{s_j\}_{j=1}^{k} \) such that \( \sup_{I_j} f - f(s_j) < \varepsilon/3(b-a) \) for each \( j = 1, \ldots, k \) and sample points \( \{t_j\}_{j=1}^{k} \) such that \( f(t_j) - \inf_{I_j} f < \varepsilon/3(b-a) \) for each \( j = 1, \ldots, k \). Then

\[
\sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j \leq \sum_{j=1}^{k} \left( \sup_{I_j} f - f(s_j) \right) \Delta_j
\]

\[
\quad + \left| \sum_{j=1}^{k} f(s_j) \Delta_j - \sum_{j=1}^{k} f(t_j) \Delta_j \right|
\]

\[
\quad + \sum_{j=1}^{k} \left( f(t_j) - \inf_{I_j} f \right) \Delta_j
\]

\[
\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\]

\[
\quad = \varepsilon
\]

\((\Leftarrow)\) In this direction, we use the **The Refinement Characterization** to show that \( f \) is Riemann integrable. Let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that whenever \( \mathcal{P} = \{a, x_1, x_2, \ldots, x_{k-1}, b\} \) is a partition of \([a, b]\) with \( m(\mathcal{P}) < \delta \) we have

\[
\sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j < \varepsilon
\]
Let \( \mathcal{P} \) and \( \mathcal{P}' \) be partitions of \([a, b]\) with \( m(\mathcal{P}) \), \( m(\mathcal{P}') < \delta \), and let \( \mathcal{Q} = \mathcal{P} \cup \mathcal{P}' \) be their common refinement. Say \( \mathcal{P} \) has \( k \) points and \( \mathcal{Q} \) has \( n \) points. Then

\[
|R(f, \mathcal{P}) - R(f, \mathcal{Q})| = \left| \sum_{j=1}^{k} f(s_j) \Delta_j - \sum_{l=1}^{n} f(t_l) \tilde{\Delta}_l \right|
\]

\[
= \left| \sum_{j=1}^{k} \left( \sum_{I_l \subseteq I_j} [f(s_j) - f(t_l)] \tilde{\Delta}_l \right) \right|
\]

\[
\leq \sum_{j=1}^{k} \left( \sum_{I_l \subseteq I_j} |f(s_j) - f(t_l)| \tilde{\Delta}_l \right)
\]

\[
\leq \sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j
\]

\[
< \varepsilon \quad \Box
\]

6) We are asked to prove or disprove the following conjecture:

**Conjecture:** If \( f : [a, b] \to \mathbb{R} \) is bounded and \( \forall \varepsilon > 0 \) there exists a partition \( \mathcal{P} = \{x_0, x_1, x_2, \ldots, x_k\} \) such that

\[
\sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j < \varepsilon,
\]

then \( f \) is integrable on \([a, b]\).

**Claim:** The conjecture is true.

**Proof:** We use the characterization of Riemann integrability provided in the previous problem. Let \( \varepsilon > 0 \). By hypothesis, \( f \) is bounded so there exists \( M \) such that \( |f(x)| \leq M \) for all \( x \in [a, b] \). Moreover, by hypothesis there exists a partition \( \mathcal{P} = \{a, x_1, x_2, \ldots, x_{k-1}, b\} \) of \([a, b]\) such that

\[
\sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_j < \frac{\varepsilon}{2},
\]
Let $\delta = \varepsilon/(8kM)$. Suppose $\mathcal{P}$ is an $n$-point partition of $[a,b]$ satisfying $m(\mathcal{P}) < \delta$, where $J_l$ denotes the $l$th subinterval of $\mathcal{P}$. We examine

$$\sum_{l=1}^{n} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l = \sum_{l: \forall j, x_j \notin J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l + \sum_{l: \exists j, x_j \in J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l$$

(1)

We examine the two sums on the right side of (1) separately. We may regroup the first sum to obtain

$$\sum_{l: \forall j, x_j \notin J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l = \sum_{j=1}^{k} \left( \sum_{J_l \subset I_j: x_j \notin J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l \right)$$

(2)

Note that all we have done here is group the subintervals $J_l$ not containing any $x_j \in \tilde{\mathcal{P}}$ according to the subintervals $I_j$ of $\tilde{\mathcal{P}}$ in which they are housed, taking $j = 1, 2, \ldots, k$ one at a time.

In this new sum on the right, however, we are able to make valuable estimates. For instance, if $J_l \subset I_j$, then $\sup_{J_l} f - \inf_{J_l} f \leq \sup_{I_j} f - \inf_{I_j} f$. Moreover, if $J_l \subset I_j$ and $x_{j-1}, x_j \notin J_l$, then $J_l$ is in the interior of $I_j$. Thus, the sum of all such corresponding $\Delta_l$ is strictly less than $\tilde{\Delta}_j$. So continuing from the right hand side of (2), we have

$$\sum_{j=1}^{k} \left( \sum_{J_l \subset I_j: x_j \notin J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l \right) \leq \sum_{j=1}^{k} \left( \sum_{J_l \subset I_j: x_{j-1}, x_j \notin J_l} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta_l \right) \leq \sum_{j=1}^{k} \left( \sup_{I_j} f - \inf_{I_j} f \right) \tilde{\Delta}_j \leq \frac{\varepsilon}{2}$$

Now, the second sum on the right of (1) is much easier and is the origin of our very strangely chosen $\delta$. After all, there are only $k$ subintervals in the partition $\tilde{\mathcal{P}}$, so the number of $J_l$ that contain an $x_j \in \tilde{\mathcal{P}}$ is at most $2k$. Moreover, for any such $l$, $\sup_{J_l} f - \inf_{J_l} f \leq 2M$ because $|f(x)| \leq M$ for all $x \in [a,b]$. Thus,

$$\sum_{l: \exists j, x_j \in J_l} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l \leq (2M)(2k)m(\mathcal{P}) < (2M)(2k)\delta = \frac{\varepsilon}{2}$$
Therefore both sums in (1) are no more than $\varepsilon/2$, so we have exhibited $\delta$ such that
\[
\sum_{l=1}^{n} \left( \sup_{J_l} f - \inf_{J_l} f \right) \Delta_l < \varepsilon
\]
whenever $m(\mathcal{P}) < \delta$. By the result of the previous problem, $f$ is Riemann integrable. $\square$