Math 143    Winter 2016    Exam I
Prof. Retsek

This examination has five (5) questions, some with multiple parts. Please look over your booklet and if it is incomplete notify me at once. Next to each part of each question you will find the relative value of that particular part. There are 50 points possible.

You must show all your work and clearly indicate your answer in order to receive full credit. Partial credit will be given for partially completed solutions. Stay calm and focused. Good luck!
1. (10 points) Answer each question carefully, no need to show any work.

(a) True or False: If \( a_n \to 0 \), then \( \sum a_n \) converges.

\[
\text{False : } \sum \frac{1}{n} \text{ diverges}
\]

(b) State the Squeeze Theorem for sequences.

If \( a_n \leq b_n \leq c_n \) for all \( n \) and
\( a_n \to L \) and \( c_n \to L \), then \( b_n \to L \).

(c) Write down a specific series with strictly positive terms that converges to \( \frac{5}{3} \).

\[
\frac{5}{6} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = \frac{5}{6} \left[ \frac{1}{1 - \frac{1}{2}} \right] = \frac{5}{3} \quad \text{(just one example, there are others)}
\]

(d) Give an example of a series that converges only conditionally.

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n}
\]

(e) Give an example of a convergent series for which Ratio Test is inconclusive.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]
2. (10 points) Determine whether the series converges absolutely, converges conditionally, or diverges. Show all of your work and justify all of your assertions.

(a) \[ \sum_{n=1}^{\infty} (-1)^n \left( \frac{2^{3n+1}}{7^n} \right) = \sum_{n=1}^{\infty} 2 \left( -\frac{8}{7} \right)^n \] a divergent geometric series since \( \left| -\frac{8}{7} \right| > 1 \).

(b) \[ \sum_{n=1}^{\infty} \frac{-n+3}{\sqrt{2n^5-n+1}} = -\left( \sum_{n=1}^{\infty} \frac{n-3}{12n^5-n+1} \right) \] * positive terms for \( n > 3 \), so LCT allowed.

Consider \( \lim_{n \to \infty} \frac{n-3}{12n^5-n+1} = \lim_{n \to \infty} \frac{n^{5/2} - 3n^{3/2}}{12n^5-n+1} \)

\[ = \lim_{n \to \infty} \frac{1 - \frac{3}{n}}{\frac{1}{\sqrt{2} - \frac{1}{n^{1/2}} + \frac{1}{n^5}}} \]

\[ = \frac{1}{\sqrt{2}} \]

Since \( \sum \frac{1}{n^{3/2}} \) converges ( \( p = \frac{3}{2} > 1 \)),

\[ \sum \frac{n-3}{12n^5-n+1} \] does too by LCT.
3. (10 points) Determine whether the series

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1 \ln n}} \]

converges absolutely, converges conditionally, or diverges. Show all of your work and justify all of your assertions.

(i) The series converges by AST since \( \frac{1}{n^{1 \ln n}} \to 0 \).

(ii) To decide AC vs. CC, we consider

\[ \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n^{1 \ln n}} \right| = \sum_{n=2}^{\infty} \frac{1}{n^{\ln n}} \]

Now, \( f(x) = \frac{1}{x^{\ln x}} \) is continuous and decreasing on \((2, \infty)\) so we may employ Integral Test.

\[ \int_{2}^{\infty} \frac{1}{x^{\ln x}} \, dx = \lim_{b \to \infty} \left( \int_{2}^{b} \frac{1}{x^{\ln x}} \, dx \right) \]

Let \( u = \ln x \), \( du = \frac{1}{x} \, dx \)

\[ \lim_{b \to \infty} \left( \int_{\ln 2}^{\ln b} \frac{1}{u} \, du \right) \]

\[ = \lim_{b \to \infty} \left( \ln b - \ln 2 \right) \]

\[ = \lim_{b \to \infty} \left( 2 \sqrt{b} - 2 \sqrt{\ln 2} \right) \]

\[ = \infty \]

\[ \sum_{n=2}^{\infty} \frac{1}{n^{\ln n}} \text{ diverges by I.T., so} \]

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1 \ln n}} \text{ is merely C.C.} \]
4. (10 points) Determine whether the series converges or diverges. Show all of your work and justify all of your assertions.

(a) \[ \sum_{n=1}^{\infty} \frac{4^n(n+1)ln}{(2n)!} \]

Employing the Ratio Test,

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{4^{n+1}(n+2)(n+1)}{(2n+1)!} \cdot n}{\frac{4^n(n+1)n}{(2n)!}} = \lim_{n \to \infty} \frac{4(n+2)(n+1)}{(2n+2)(2n+1)n} = \lim_{n \to \infty} \frac{4(n^2 + 3n + 2)}{4n^3 + 6n^2 + 2n} = 0 \]

\[ = 0 < 1 \]

converge by Ratio Test

(b) \[ \sum_{n=1}^{\infty} \frac{3^n n^2}{n^{2n}} \]

Employing the Root Test,

\[ \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{3^n}{n^2} = \infty \]

\[ \text{L'Hopital's rule is applied twice on} \]

\[ \lim_{x \to \infty} \frac{3^x}{x^2} = \lim_{x \to \infty} \frac{3^x \ln 3}{2x} = \lim_{x \to \infty} \frac{3^x \ln 3 \ln 3}{2} = \infty \]

= \infty

diverge by Root Test
5. (10 points) Consider the series

\[ \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \]

It's easy to show it converges by Ratio Test. In this problem we actually find the sum!?

(a) Let \( S_N \) stand for the sum of the first \( N \) terms. By hand, calculate \( S_1 \) and \( S_2 \).

\[ S_1 = \frac{1}{2!} = \frac{1}{2}, \quad S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6} \]

(b) Notice from part (a) that \( S_1 = \frac{(1+1)! - 1}{(1+1)!} \) and \( S_2 = \frac{(2+1)! - 1}{(2+1)!} \). Without calculating, guess \( S_3 \).

**Guess:** \( S_3 = \frac{(3+1)! - 1}{(3+1)!} = \frac{23}{24} \)

(c) Verify your guess from (b). [Hint: \( S_3 \) is just \( S_2 + a_3 \), right?]

\[ S_3 = S_2 + a_3 = \frac{5}{6} + \frac{3}{(3+1)!} = \frac{5}{6} + \frac{3}{24} = \frac{23}{24} \]

(d) Guess a formula for \( S_N \) and use your guess to find the sum of the infinite series.

**Guess:** \( S_N = \frac{(N+1)! - 1}{(N+1)!} \)

If so, \( \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{(N+1)! - 1}{(N+1)!} = \lim_{N \to \infty} \frac{1 - \frac{1}{(N+1)!}}{1} = 1 \)

\[ \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1 \]