(1) (20 points) Rewrite the integral
\[ \int_{-1}^{1} \int_{x^2}^{1-x} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx \]
in the orders of iteration \( dz \, dx \, dy \) and \( dx \, dy \, dz \).
(2) Let \( \mathcal{R} \) be the semicircular region bounded by \( y = \sqrt{1-x^2} \) and the \( x \)-axis.

(14 points) (a) Find the center of mass of the region \( \mathcal{R} \) assuming density function
\[
\rho(x, y) = (x^2 + y^2)^{\frac{n}{2}}.
\]

[Suggestion: Argue succinctly that one coordinate of \((\bar{x}, \bar{y})\) is clearly zero.]

\[
\begin{align*}
\bar{y} &= \frac{1}{m} \iiint_{\mathcal{R}} y \rho(x,y) \, dA \\
&= \frac{1}{m} \int_0^1 \int_0^{\pi/2} r^2 \sin \theta (r^2)^{\frac{n}{2}} r \, dr \, d\theta \\
&= \frac{1}{m} \int_0^{\pi/2} \sin \theta (r^2)^{\frac{n}{2}} r \, dr \, d\theta \\
&= \frac{1}{m} \left( \frac{n+2}{\pi} \right) \left( \frac{2}{n+3} \right) \\
&= \frac{2}{\pi} \left( \frac{n+2}{n+3} \right)
\end{align*}
\]

(6 points) (b) Discuss the migration of \((\bar{x}, \bar{y})\) as \( n \to \infty \). Why does this migration make physical sense?

As \( n \to \infty \), \( \bar{y} \to \frac{2}{\pi} \). It makes sense that \( \bar{y} \) would migrate "upward" since the density is \( \rho = r^n \to \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r = 1 \end{cases} \). It "settles" at \( \bar{y} = \frac{2}{\pi} \) (rather than \( \bar{y} \to 1 \), say) because the "rim" of the region remains constant density \( 1 \) as \( n \to \infty \) while the interior gets less and less dense. The value \( \bar{y} = \frac{2}{\pi} \) is the balancing point for the rim that remains.
(3) (20 points) Calculate

\[ \int_{0}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy \]

Rewriting in spherical coordinates, we obtain

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq \varphi \leq \pi \]
\[ 0 \leq \rho \leq 2 \]

\[ \int_{0}^{\pi/2} \int_{0}^{\pi} \int_{0}^{2} \sqrt{\rho^2} \, \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \]

\[ = \int_{0}^{\pi/2} \int_{0}^{\pi} \int_{0}^{2} \rho^3 \, \sin \varphi \, d\rho \, d\varphi \, d\theta \]

\[ = 4 \int_{0}^{\pi/2} \int_{0}^{\pi} \sin \varphi \, d\varphi \, d\theta \]

\[ = 8 \int_{0}^{\pi/2} \, d\theta \]

\[ = 4 \pi \]
(4) (5 points) (a) Sketch two water pipes of radius one foot that intersect perpendicularly.

(15 points) (b) How much water can possibly be in both pipes at once?

[Suggestion: Focus on calculating a sensible $1/16$ of the volume common to both pipes.]

Thus, $V = 16 \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} 1 \, dz \, dx \, dy$

$= 16 \int_0^1 \int_0^y \sqrt{1-y^2} \, dx \, dy$

$= 16 \int_0^1 y \sqrt{1-y^2} \, dy$

$= 16 \left[ -\frac{1}{3} (1-y^2)^{3/2} \right]_{y=0}^{y=1}$

$= \frac{16}{3}$
(5) (5 points) (a) Sketch the region $R$ in the first quadrant of the $xy$-plane consisting of all points $(x, y)$ satisfying $1 \leq x + y \leq 2$.

![Diagram of region $R$](image)

(5 points) (b) Sketch the region $S$ in the $uv$-plane that $R$ comes from under the transformation $x = \frac{1}{2}(u - v), y = \frac{1}{2}(u + v)$.

Since $u = x + y \neq v = y - x$, $R$ came from:

![Diagram of region $S$](image)

(10 points) (c) Compute

$$\iint_{R} e^{\frac{y-x}{y+x}} \, dA.$$ 

$$\iint_{S} e^{\frac{v}{u}} \, dA \quad \frac{J}{\begin{vmatrix} \frac{2x}{2u} \frac{2y}{2v} \\ \frac{2x}{2v} \frac{2y}{2u} \end{vmatrix}} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} \cdot \frac{1}{2}$$

$$\iint_{S} e^{\frac{v}{u}} \, dA = \int \int e^{\frac{v}{u}} \left( \frac{1}{2} \right) \, dA$$

$$= \frac{1}{2} \int_{1}^{2} \int_{-u}^{v} e^{\frac{v}{u}} \, dv \, du$$

$$= \frac{1}{2} \left[ \int_{1}^{2} u e^{\frac{v}{u}} \right]_{v=-u}^{v=u} \, du$$

$$= \frac{1}{2} \int_{1}^{2} u (e - \frac{1}{2}) \, du$$

$$= \frac{1}{4} (e - \frac{1}{2}) \left[ u^2 \right]_{u=1}^{u=2}$$

$$= \frac{3}{4} (e - \frac{1}{2})$$