A Glimpse of Fourier Analysis

Dylan Q. Retsek

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A Note to the Student

These notes were written to give us a look into the subject of Fourier Analysis. One could take a year long course in the subject, indeed even write a PhD thesis. We have four weeks. Our goal, therefore, will be to kindle an interest (or at least respect for) the theoretical underpinnings and key questions in the elementary study of Fourier series.

Nearly every structural object in these notes requires action.

**Definition.** Definitions are the key objects in these pages. They are agreements upon the meaning of terms or phrases. The defined term or phrase is always in emphasized text. These agreements are inviolable and no defined term is ever used loosely. The required actions are memorization, digestion and contextualization.

**Example.** Examples are provided to highlight or lend interest to a new term or idea and generate discussion. Required actions are digestion and contextualization.

**Exercise.** Exercises are explicit actions requested of the reader, usually in order to put new terms or ideas to use, test comprehension, and generate discussion.

**Proposition.** Propositions are statements that may or may not be true. They depend only on definitions, propositions and theorems that have come before them in the notes. The first required action is to assess whether the statement is true or false. The remaining required action depends on your assessment. If true, prove it; if false, provide a counterexample to the statement.

**Theorem.** Theorems are statements that are true and are valuable because of wide applicability or deep consequence. They depend only on definitions, propositions and theorems that have come before them in the notes. The required action is to prove the theorem.

Make no mistake, this is a golden opportunity. You have earned this privilege through dedication to analysis, relish in it!

*San Luis Obispo, 2014*
Chapter 1

An Alternative to Taylor Series

1.1 Great Expectations

We wish to study series that use sines and cosines, specifically \( \cos n\theta \) and \( \sin n\theta \), as the building blocks in the same way that we have already studied Taylor series. The basic question is: what happens when we use trigonometric functions instead of powers of \( x \)? Because the sine and cosine functions are periodic, we restrict our attention to so-called \( 2\pi \)-periodic functions.

**Definition 1.1.** A function \( f : \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic if \( f(x + 2\pi) = f(x) \) for all \( x \in \mathbb{R} \).

**Exercise 1.2.** Given \( p \in \mathbb{R} \), formulate a definition of \( p \)-periodic and show that we can convert any \( p \)-periodic function into a \( 2\pi \)-periodic function by an appropriate change of variable.

We will initially restrict our attention to functions that are “piecewise continuous”.

**Definition 1.3.** A function \( f : [a, b] \to \mathbb{R} \) is piecewise continuous if it is continuous at all but finitely many points and at each discontinuity the left- and right-hand limits exist.

**Exercise 1.4.** Formulate a definition for a \( 2\pi \)-periodic piecewise continuous function \( f : \mathbb{R} \to \mathbb{R} \).

In order to expand functions as infinite linear combinations of sines and cosines, we shall encounter expressions of the form

\[
(1.1) \quad f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{i n \theta}
\]
where we steal notation from complex analysis:

\[ e^{in\theta} = \cos n\theta + i \sin n\theta. \]

**Definition 1.5.** The formal expression \( \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \) is interpreted as the limit of the symmetric partial sums:

\[ \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{in\theta}. \]

Since our goal is to expand functions in sines and cosines, it is instructive to see the sum in Definition (1.5) expanded in trigonometric functions.

**Exercise 1.6.** Write out the sum

\[ \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \]

in sines and cosines over non-negative indices \( n \).

Many questions lie ahead, but of primary importance at present is: When can we expect a \( 2\pi \)-periodic function to be expressible as in Equation (1.1)? To start to get our hands on this question, we examine the nature of the coefficients in the case where \( f \) really can be expressed in this way.

**Exercise 1.7.** Assuming the \( 2\pi \)-periodic function \( f \) can be expressed as in Equation (1.1), show that the form of the coefficients \( c_n \) is given by

\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta. \]

[Assume whatever ideal conditions you like on \( f \), especially regarding term-by-term integration of the series.]

As always, we could instead choose to focus on the expansion in terms of sines and cosines, rather than exponentials. In this case, the coefficients have integral formulas analogous to that in (1.2).

**Exercise 1.8.** Assuming the \( 2\pi \)-periodic function \( f \) can be expressed as in Equation (1.1), determine the integral form of the coefficients \( a_n \) and \( b_n \) on \( \cos n\theta \) and \( \sin n\theta \), respectively.
Well, so what? It seems at the moment that all we have is a bunch of *ifs*. *If* $f$ is already of the form (1.1), and *if* term-by-term integration of the series is legitimate, and *if* the integral in 1.2 is even defined, *then* we know the form of the coefficients $c_n$.

Be that as it may, we can take the cue from all these *ifs* and define the Fourier coefficients by the formulas developed above.

**Definition 1.9.** Given a $2\pi$-periodic function on $\mathbb{R}$, define the *Fourier coefficients* of $f$ by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta,$$

and the *Fourier series* of $f$ by

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

In order to make this all more concrete, we now consider a couple of examples.

**Example 1.10.** Let $f : \mathbb{R} \to \mathbb{R}$ be the $2\pi$-periodic function determined by

$$f(\theta) = \theta \quad (-\pi < \theta \leq \pi).$$

Calculate the Fourier series of $f$.

**Exercise 1.11.** Grouping the $n$th and $-n$th terms together in the series of Example 1.10, write the Fourier series for $f$ using only indices $n = 1, 2, 3, \ldots$.

Now take care, just because we can compute the coefficients $c_n$ and build the corresponding Fourier series of $f$, we have no reason a priori to believe that the Fourier series converges, let alone back to the function $f$.

In fact, the convergence of the series in Exercise 1.11 is not obvious. Toward a careful analysis, the following lemma allows for control of partial sums of sine functions.

**Lemma 1.12.** If $\theta$ is not an integer multiple of $2\pi$, then

$$\sum_{n=1}^{k} \sin(n\theta) = \frac{\sin \left( \frac{1}{2}(k+1)\theta \right) \cdot \sin \left( \frac{1}{2}k\theta \right)}{\sin \left( \frac{1}{2}\theta \right)}.$$

As a corollary to Lemma 1.12 we see that partial sums of series of sines are bounded.
Proposition 1.13. For each fixed $\theta \in \mathbb{R}$, the partial sum 
\[ \sum_{n=1}^{k} \sin(n\theta) \]
is bounded independent of $k$.

Now that we have control over the partial sums of the sine series, we can return to convergence of the series we constructed in Exercise 1.11. Consider going deep in the Math 412 bag to employ Abel’s Test for numerical series.

Proposition 1.14. The Fourier series of $f$ in Exercise 1.11 converges for each $\theta$.

Exercise 1.15. Ask a machine to plot a few partial sums of the Fourier series of $f$ in Exercise 1.11. Discuss in light of your work on Proposition 1.14.

We follow a similar path on the next example.

Example 1.16. Let $g : \mathbb{R} \to \mathbb{R}$ be the $2\pi$-periodic function determined by 
\[ g(\theta) = |\theta| \quad (-\pi \leq \theta \leq \pi). \]
Calculate the Fourier series of $g$.

Exercise 1.17. Write the Fourier series for $g$ using only nonnegative indices.

Proposition 1.18. The Fourier series of $g$ in Exercise 1.17 converges uniformly and absolutely.

Exercise 1.19. Ask a machine to plot a few partial sums of the Fourier series of $g$ in Exercise 1.17. Discuss in light of your work on Proposition 1.18.

1.2 Does It Always Work?

In Section 1.1 we convinced ourselves that if 
\[ f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \]
then the Fourier coefficients $c_n$ are determined by $f$. We then turned this idea around and defined the Fourier coefficients for any $2\pi$-periodic function $f$ by 
\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} d\theta. \]
So the big questions now are, under what conditions on \( f \) do these \( c_n \) make sense, does the resulting Fourier series converge, and does the resulting Fourier series converge back to the function \( f \)? (These all appear to be affirmative in Examples 1.10 and 1.16, but generally?)

**Proposition 1.20.** If \( f \) is \( 2\pi \)-periodic and piecewise-continuous on \( \mathbb{R} \), then the Fourier coefficients \( c_n \) exist for each \( n \in \mathbb{N} \).

**Proposition 1.21.** If \( f \) is \( 2\pi \)-periodic and piecewise-continuous on \( \mathbb{R} \), then the sequence of Fourier coefficients \( \{c_n\} \) is bounded.

**Proposition 1.22.** If \( \{c_n\} \) is bounded, then the Fourier series \( \sum c_n e^{in\theta} \) necessarily converges.

The preceding propositions show that if we hope to show that Fourier series converge, we need stricter control over the coefficients \( c_n \). In order to exact that control, we need to briefly digress on the size of complex numbers in general.

A complex number \( z = a + bi \) can be thought of as being a point in the “complex plane” at pythagorean distance \( \sqrt{a^2 + b^2} \) from the origin. So we define the magnitude of \( z \) by \( |z| = \sqrt{a^2 + b^2} \). The conjugate of \( z = a + bi \) is the complex number \( \bar{z} = a - bi \). That \( |z|^2 = z\bar{z} \) (check this by foiling!) is our gateway observation to greater control on the coefficients \( c_n \).

**Proposition 1.23.** If \( f \) is \( 2\pi \)-periodic and piecewise-continuous on \( \mathbb{R} \), then
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{n=-N}^{N} c_n e^{in\theta} \right|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{n=-N}^{N} |c_n|^2.
\]

**Corollary 1.24.** If \( f \) is \( 2\pi \)-periodic and piecewise-continuous on \( \mathbb{R} \), then
\[
\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.
\]

**Corollary 1.25.** If \( f \) is \( 2\pi \)-periodic and piecewise-continuous on \( \mathbb{R} \), then \( \lim_{n \to \pm \infty} c_n = 0 \).
We have exacted strict control on the Fourier coefficients $c_n$. We are thus ready to return to the fundamental question of the present section: under what conditions will the Fourier series for $f$ converge back to $f$, or converge at all, and to what? This is clearly a question of partial sums of the Fourier series, which we may as well adorn with some notation:

**Definition 1.26.** Given a function $f$, let

$$S^f_N(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) e^{-in\psi} d\psi.$$ 

Because $c_n$ is given as an integral, it is possible to express $S^f_N(\theta)$ as an integral itself.

**Proposition 1.27.** Given a $2\pi$-periodic and piecewise-continuous function $f$,

$$S^f_N(\theta) = \int_{-\pi}^{\pi} f(\varphi + \theta) D_N(\varphi) d\varphi$$

where

$$D_N(\varphi) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\varphi}.$$ 

In words, Proposition 1.27 says that the partial sums of the Fourier series for a function $f$ can be recovered by integrating $f$ against a certain function $D_N$. This is sometimes called a *convolution* (we saw this idea fleetingly in Math 413, do you remember in what context?). The function $D_N$ has a special name after one of your mathematical ancestors.

**Definition 1.28.** The function $D_N(\varphi)$ is called the *Nth Dirichlet kernel*.

Just as in Math 413, the special features of the kernel are what translate into information about the convolution itself, and, by Proposition 1.27, ultimately into the partial sums of the Fourier series. We begin to investigate $D_N(\varphi)$ with a couple of calculations.

**Exercise 1.29.** Calculate $\int_{-\pi}^{0} D_N(\varphi) d\varphi$ and $\int_{0}^{\pi} D_N(\varphi) d\varphi$.

As we have seen, it may behoove us to have certain sums written out in “closed form” (without $\sum$’s). This is certainly possible for the Dirichlet kernel by exploiting its relation to a geometric sum.
Proposition 1.30. The $N$th Dirichlet kernel is
\[ D_N(\varphi) = \frac{1}{2\pi} e^{i(N+1)\varphi} - e^{-iN\varphi} = \frac{\sin(N + \frac{1}{2})\varphi}{2\pi \sin \frac{\varphi}{2}}. \]

Exercise 1.31. Ask a machine to plot $D_N(\varphi)$ for a few values of $N$. Discuss in light of your work on Proposition 1.30.

As suggested earlier, it is the Dirichlet kernel and its attendant properties that will provide the first path to convergence of Fourier series in the sequel. Section 1.3 provides our first big theorem along these lines.

1.3 A Big Theorem

Our first big theorem requires a bit more than piecewise continuity of our $2\pi$-periodic functions.

Definition 1.32. The function $f : [a, b] \to \mathbb{R}$ is piecewise smooth if it is $C^1$ at all but finitely many points and the four one-sided limits
\[ f(\theta-) = \lim_{\varepsilon \to \theta^-} f(\varepsilon) \quad f(\theta+) = \lim_{\varepsilon \to \theta^+} f(\varepsilon) \]
\[ f'(\theta-) = \lim_{\varepsilon \to \theta^-} f'(\varepsilon) \quad f'(\theta+) = \lim_{\varepsilon \to \theta^+} f'(\varepsilon) \]
exist at every point.

Exercise 1.33. Sketch a graph of a continuous function on $[a, b]$ that is not piecewise smooth.

We now have the terminology to state our objective in this section: we wish to show that for $2\pi$-periodic, piecewise smooth $f$, the Fourier series converges pointwise at every $\theta$. We have seen in Example 1.10 that the Fourier series doesn’t necessarily converge back to $f$ (there, the issue was suggestively at the jump discontinuities of the original function), but we are undeterred.

As a first step in the right direction, let us examine a “sort of derivative” of the typical piecewise smooth function $f$.

Exercise 1.34. Suppose $f$ is piecewise smooth and $2\pi$-periodic and let $g_{\theta} : [-\pi, 0) \to \mathbb{C}$ be given by
\[ g_{\theta}(\varphi) = \frac{f(\varphi + \theta) - f(\varphi - \theta)}{e^{i\varphi} - 1}. \]
Calculate $\lim_{\varphi \to 0^-} g_{\theta}(\varphi)$. 
Exercise 1.35. By analogy, formulate and carry out an exercise like Exercise 1.34 only for the appropriate right-hand limit.

We next directly examine the difference between $S_N^f$ and the "right and left hand average" $\frac{1}{2}[f(\theta-) + f(\theta+)]$ for a piecewise smooth function $f$. Looking back at Examples 1.10 and 1.16, is this average a good guess for the limit of the Fourier series?

Proposition 1.36. Suppose $f$ is $2\pi$-periodic and piecewise smooth. Then for each $\theta \in \mathbb{R}$,

\begin{equation}
S_N^f(\theta) - \frac{1}{2}[f(\theta-) + f(\theta+)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\varphi) [e^{i(N+1)\varphi} - e^{-iN\varphi}] d\varphi,
\end{equation}

where the function $G$ is given by

\[
G(\varphi) = \begin{cases} 
g_\theta^- & \text{if } -\pi \leq \varphi < 0 \\
414 & \text{if } \varphi = 0 \\
g_\theta^+ & \text{if } 0 < \varphi \leq \pi \end{cases}.
\]

While not the prettiest function we’ve ever seen, the function $G$ of Proposition 1.36 is a perfectly suited means to our desired end.

Proposition 1.37. The function $G$ in (1.3) is piecewise continuous on $[-\pi, \pi]$.

At long last, we are now prepared to put the puzzle pieces together and prove our big theorem. The key here is that we wish to show that the left-hand side of equality (1.3) tends to zero as $N \to \infty$, and we do it through consideration of the right-hand side. Here’s the big theorem:

Theorem 1.38. Suppose $f$ is $2\pi$-periodic and piecewise smooth. Then the Fourier series of $f$ converges pointwise to the function $\frac{1}{2}[f(\theta-) + f(\theta+)]$.

Corollary 1.39. If $f$ is $2\pi$-periodic, piecewise smooth and continuous, then

\[ f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}. \]

We conclude this section with some fun and interesting consequences of our big theorem. The first is an alternate route to a fact we already know from Math 413.
Exercise 1.40. Apply Theorem 1.38 to Example 1.10 at the point $\theta = \frac{\pi}{2}$. What do you see and what was our route to the same fact in Math 413?

Here’s another interesting sum...

Exercise 1.41. Apply Corollary 1.39 to Example 1.16 at the point $\theta = 0$.

And finally, another familiar friend from a whole new approach:

Proposition 1.42. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
Chapter 2

Summability Techniques

In Chapter 1 we showed that under reasonable conditions on a function $f$, the Fourier series of $f$ converges back to $f$ (and converges anyway to $(f(x-) + f(x+))/2$ at points where $f$ is not even continuous). Though our starting assumptions on $f$ are relatively lax, especially compared to the $C^\infty$ requirement on $f$ to even build the Taylor series, we might prefer to get away without requiring piecewise smoothness.

In this chapter, we step back and turn our attention to the following tantalizing question:

**Question:** Provided we can build the sequence of Fourier coefficients $\{c_n\}$, might there be other methods to recover the original function $f$ from these data?

Indeed, nobody said we had to use partial summation to get back to $f$. It is eye-opening and liberating to think that we could employ the $\{c_n\}$ in any way we see fit in order to recover the function $f$. This chapter is about two such options.

### 2.1 Fejér Summability

In Math 412 we considered the following homework exercise.

**Exercise 2.1.** If the sequence (of numbers) $a_j \to \alpha$, then the sequence of averages $\{b_j\}$ given by

$$b_j = \frac{a_1 + a_2 + \cdots + a_j}{j}$$

also converges to $\alpha$. Moreover, there are divergent sequences $\{a_j\}$ for which the sequence of averages $\{b_j\}$ converges.
The point of Exercise 2.1 is that “average sequences converge to the same thing, and sometimes behave even better than the original.” This is the key idea of Fejér’s method; since the sequence of partial sums $S_N^f(\theta)$ might not converge back to $f(\theta)$, why not consider the associated sequence of averages?

**Definition 2.2.** Given the function $f$ with Fourier coefficients $\{c_n\}$, define

$$
\sigma_N^f(\theta) = \frac{S_0^f(\theta) + S_1^f(\theta) + S_2^f(\theta) + \cdots + S_N^f(\theta)}{N + 1}.
$$

We know it is possible that $S_N^f(\theta) \not\to f(\theta)$, but maybe we have $\sigma_N^f(\theta) \to f(\theta)$ in these cases? And if not always, when? Following our consideration of the Dirichlet kernel in Chapter 1, we here develop the analogous kernel for averages of partial sums.

**Proposition 2.3.** Given a $2\pi$-periodic and piecewise-continuous function $f$,

$$
\sigma_N^f(\theta) = \int_{-\pi}^{\pi} f(\varphi + \theta) K_N(\varphi) \, d\varphi
$$

where

$$
K_N(\varphi) = \frac{1}{2\pi} \sum_{n=-N}^{N} \frac{N + 1 - |n|}{N + 1} e^{in\varphi}.
$$

In words, Proposition 2.3 says that the averages of the partial sums of the Fourier series for a function $f$ can be recovered by integrating $f$ against a certain function $K_N$. This is another instance of the idea of convolution. The function $K_N$ has a special name after the nineteen year old who discovered it (no lie).

**Definition 2.4.** The function $K_N(\varphi)$ is called the $N$th Fejér kernel.

Just as in Chapter 1, the special features of the kernel are what translate into information about the convolution itself, and, by Proposition 2.3, ultimately into the averages of the partial sums of the Fourier series. We begin to investigate $K_N(\varphi)$ with a couple of calculations.

**Exercise 2.5.** Calculate $\int_{-\pi}^{0} K_N(\varphi) \, d\varphi$ and $\int_{0}^{\pi} K_N(\varphi) \, d\varphi$.

As we have seen, it may behoove us to have certain sums written out in “closed form” (without $\sum$’s). This is certainly possible for the Fejér kernel (as for the Dirichlet kernel) by exploiting its relation to a geometric sum.
Proposition 2.6. The $N$th Fejér kernel is

$$K_N(\varphi) = \frac{1}{2(N+1)\pi} \left( \frac{\sin \left( \frac{(N+1)\varphi}{2} \right)}{\sin \frac{\varphi}{2}} \right)^2.$$ 

when $\varphi \neq 0$ and $K_N(0) = (N+1)/2\pi$.

Exercise 2.7. Ask a machine to plot $K_N(\varphi)$ for a few values of $N$. Discuss in light of your work on Proposition 2.6.

From Exercises 2.5 and 2.7 we see certain similarities between the Dirichlet and Fejér kernels. The next lemma formalizes one such property of the Fejér kernel that will be pivotal in our next big theorem.

Lemma 2.8. For each $\delta > 0$, the Fejér kernel $K_N \to 0$ uniformly outside $[-\delta, \delta]$.

In words, Lemma 2.8 says that when $N$ is large “most” of $K_N$ is concentrated near the origin (see those images from Exercise 2.7). Thus, if $f$ is continuous at $\theta$ Equation (2.1) implies that $\sigma_N^f(\theta) \approx f(\theta)$. Formalizing these ideas, we have our next big theorem.

Theorem 2.9. If $f$ is $2\pi$-periodic and piecewise-continuous, then

$$\sigma_N^f(\theta) \to f(\theta) \quad \text{as} \quad N \to \infty$$

at each point $\theta$ at which $f$ is continuous.
Bibliography