Homework # 2 Solutions, Math 412

1) We defined the relation “<” on the set \( \mathcal{F} \) of all cuts by

\[ A < B \iff A \subsetneq B. \]

In order to show that “<” is an ordering on the set of all cuts, we must show two things. First, we must argue that “<” is transitive; this is a non-issue, though. If \( A \subsetneq B \) and \( B \subsetneq C \), then \( A \subsetneq C \). This is true of any sets and has nothing in particular to do with cuts.

The second, and more involved requirement is that given two cuts \( A \) and \( B \), exactly one of

\[ A \subsetneq B, A = B, B \subsetneq A \]

holds true. Certainly no more than one of these things can be true at once for any sets \( A \) and \( B \), never mind cuts. To show at least one holds, suppose the first two don’t hold. In this case, we claim that \( B \subsetneq A \).

To wit, let \( w \in B \). Since neither \( A \subsetneq B \) nor \( A = B \) holds, there must exist \( z \in A \setminus B \). Now, if \( w > z \) then \( z \in B \) since \( B \) is a cut (property (ii) of cuts). So \( w \) is not greater than \( z \). Of course, \( w \neq z \) either since \( z \) is not in \( B \). Thus, it must be that \( w < z \) and therefore \( w \in A \) since \( A \) is a cut (property (ii) again).

Since every element of \( B \) is an element of \( A \), \( B \subsetneq A \) holds. Therefore, exactly one of

\[ A \subsetneq B, A = B, B \subsetneq A \]

holds true and “<” is indeed an ordering. \( \square \)

2) Let \( x \) and \( y \) be real, say with \( x < y \). Then \( x/\sqrt{3} \) and \( y/\sqrt{3} \) are real numbers with \( x/\sqrt{3} < y/\sqrt{3} \), so there exists a rational number \( q \) between them. That is,

\[ x/\sqrt{3} < q < y/\sqrt{3} \]

But this last string of inequalities implies that

\[ x < q\sqrt{3} < y \]

I will leave it for you to convince yourself that \( q\sqrt{3} \) is irrational since the product of a rational and an irrational cannot be rational. \( \square \)
3) Here we are asked to follow up on our proof that there exist square roots in \( \mathbb{R} \) by showing that for the \( y \) and \( \epsilon \) introduced in class, if \( y^2 > x \) then \( y - \epsilon \) is an upper bound for \( A \).

Assume \( y^2 > x \). Then

\[
(y - \epsilon)^2 - x = y^2 - 2y\epsilon + \epsilon^2
\]

\[
= y^2 - 2y \left( \frac{y^2 - x}{4(x + 1)} \right) + \left( \frac{y^2 - x}{4(x + 1)} \right)^2 - x
\]

\[
= (y^2 - x) \left( 1 - \frac{y}{2(x + 1)} \right) + \left( \frac{y^2 - x}{4(x + 1)} \right)^2
\]

\[
\geq 0.
\]

Note that the equalities follow from algebra and the inequality follows because \( x + 1 \geq y \).

Now, since \( a^2 < x \) for every \( a \in A \) and we just showed that \( x \leq (y - \epsilon)^2 \), we have \( a^2 < (y - \epsilon)^2 \) for every \( a \in A \). If \( y - \epsilon > 0 \) (it is, check this!), we can square root both sides and finally obtain \( a < y - \epsilon \) for every \( a \in A \). Thus, \( y - \epsilon \) is an upper bound for \( A \), contradicting \( y = \sup(A) \).

4) In this problem we are given that \( x_j \to x \) and \( x < y \). We wish to show that “eventually” \( x_j < y \). More precisely, we wish to prove that there exists a natural number \( N \) such that \( j \geq N \) implies \( x_j < y \). Well, since \( x < y \) we know that \( y - x > 0 \). So, by virtue of \( x_j \to x \), there exists \( N \) such that \( j \geq N \) implies \( |x_j - x| < (y - x)/3 \).

Draw a diagram of a real number line. If \( x_j \) is within \( (y - x)/3 \) of \( x \), then surely \( x_j < y \). This completes the proof.

5) The first step to success on this problem is understanding what is meant by the sequence \( \{a_j\} \). We are given a sequence \( \{x_k\} \) that converges to \( x \). We then construct the sequence \( \{a_j\} \) as follows: \( a_j \) is the average of the first \( j \) terms of the original sequence \( \{x_k\} \). For example,

\[
a_1 = x_1,
\]

\[
a_2 = \frac{x_1 + x_2}{2},
\]

\[
a_3 = \frac{x_1 + x_2 + x_3}{3},
\]


and in general,
\[ a_j = \frac{x_1 + x_2 + \ldots + x_j}{j} \]

Now, if the terms of the sequence \( \{x_k\} \) are “eventually” close to \( x \), then, in the long run, we expect the averages to get close to \( x \) (the first finitely many terms, no matter how far from \( x \), should be negligible in the long run). So we prove that \( \{a_j\} \) converges to \( x \).

Let \( \varepsilon > 0 \). Choose \( N_1 \) such that \( k \geq N_1 \) implies \( |x_k - x| < \varepsilon/2 \). Now consider the sum

\[ |x_1 - x| + |x_2 - x| + |x_3 - x| + \ldots + |x_{N_1-1} - x| \]

of the absolute value of the first \( N_1 - 1 \) distances to \( x \). However big \( N_1 \) is, this sum is just some finite positive number. By the Archimedean property of the real numbers we can find \( N_2 \) such that

\[ N_2 > \frac{2(|x_1 - x| + |x_2 - x| + |x_3 - x| + \ldots + |x_{N_1-1} - x|)}{\varepsilon} \]

Set \( N = \max\{N_1, N_2\} \).

Now, let \( j \geq N \). Then

\[ |a_j - x| = \left| \frac{x_1 + x_2 + \ldots + x_j}{j} - x \right| \]
\[ = \left| \frac{x_1 + x_2 + \ldots + x_j - jx}{j} \right| \]
\[ = \left| \frac{(x_1 - x) + (x_2 - x) + \ldots + (x_j - x)}{j} \right| \]
\[ \leq \frac{|x_1 - x| + |x_2 - x| + \ldots + |x_j - x|}{j} \]
\[ = \frac{|x_1 - x| + |x_2 - x| + \ldots + |x_{N_1-1} - x|}{j} + \frac{|x_{N_1} - x| + |x_{N_1+1} - x| + \ldots + |x_j - x|}{j} \]

Focus on the first term of the last line above. Since we have chosen \( N_2 \) such that

\[ \frac{|x_1 - x| + |x_2 - x| + |x_3 - x| + \ldots + |x_{N_1-1} - x|}{N_2} < \frac{\varepsilon}{2} \]
and since $j \geq N \geq N_2$ we have

$$\frac{|x_1-x| + |x_2-x| + |x_3-x| + \ldots + |x_{N-1}-x|}{j} < \frac{\varepsilon}{2}$$

Thus the first term of (1) above is less than $\varepsilon/2$. Moreover, the second term of (1) has fewer than $j$ items in its numerator ($j-N_1-1$ to be precise), each of which is less than $\varepsilon/2$ since the subscripts on the $x$'s are all greater than $N_1$, so the entire numerator of this part is less than $j\varepsilon/2$. Since this numerator is all over $j$, the entire second term of (1) is also less than $\varepsilon/2$.

Thus, equation (1) finally yields $|a_j - x| < \varepsilon$ whenever $j \geq N$. This completes the proof.  

6) In this exercise we are first asked to provide the proof for part (2) of Proposition 2.6 in RAF.

Suppose $\{a_j\}$ converges to $\alpha$ and $\{b_j\}$ converges to $\beta$. We must show that $\{a_j+b_j\}$ converges to $\alpha+\beta$.

Let $\varepsilon > 0$. Then there exists $N_1$ such that for $j \geq N_1$ we have $|a_j - \alpha| < \varepsilon/2$ and $N_2$ such that for $j \geq N_2$ we have $|b_j - \beta| < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$.

Then for $j \geq N$,

$$|(a_j+b_j)-(\alpha+\beta)| = |(a_j-\alpha)+(b_j-\beta)| \leq |a_j-\alpha| + |b_j-\beta| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This completes the proof.  

In the next part of this exercise we are asked to provide the proof for part (4) of Proposition 2.6 in RAF.

We must show that for the sequences $\{a_j\}$ and $\{b_j\}$ above, the new sequence $\{a_j/b_j\}$ converges to $\alpha/\beta$. Clearly, for this statement to make sense we must further assume that $b_j \neq 0$ for all $j$ and $\beta \neq 0$.

Because we have $b_j$'s in the denominators of the expressions involved, we want to at once rule out the possibility that these $b_j$'s are getting arbitrarily close to zero (which could conceivably make our fractions blow up to infinity). Since $b_j \to \beta$ and $\beta \neq 0$, we can choose some $m > 0$ such that $|b_j| > m$ for all $j$ (draw a picture!!).

Next observe that if $\alpha = 0$, then $\{a_j/b_j\}$ is a list of fractions whose numerators approach 0 and whose denominators satisfy $m < |b_j| < M$ for
some $M$. So in this case, $\{a_j/b_j\}$ would converge to 0 as well (You should take care of the details here). The point is that in the rest of the proof we will assume that $\alpha \neq 0$.

Let $\varepsilon > 0$. Choose $N_1$ such that $j \geq N_1$ implies that $|a_j - \alpha| < \varepsilon m/2$ and choose $N_2$ such that $j \geq N_2$ implies that $|b_j - \beta| < \varepsilon m|\beta|/2|\alpha|$. Set $N = \max\{N_1, N_2\}$.

Then for $j \geq N$ we have

$$
\left| \frac{a_j}{b_j} - \frac{\alpha}{\beta} \right| = \left| \frac{a_j\beta - \alpha b_j}{b_j \beta} \right| = \left| \frac{(a_j - \alpha)\beta + \alpha(b_j - \beta)}{b_j \beta} \right| < \left| \frac{a_j - \alpha}{{\beta}} \right| \cdot |\beta| + |\alpha| \cdot |\beta - b_j| \left| \frac{m|\beta|}{m|\beta|} \right| < \varepsilon^{m|\beta|/2|\alpha|} = \varepsilon
$$

I completely realize that this last display may look like some kind of miracle. But remember, we did a lot of work on scratch paper to figure out how to choose $N_1$ and $N_2$ so that the miracle occurred. If you try to force $\{a_j/b_j\} - \alpha/\beta$ to be small, you will have to get a common denominator and break it up using the triangle inequality just as we did above. The rest are technical details that must be practiced. This completes the proof. □

7) Some of you wondered throughout the week if the fact that your proof for this exercise was so short meant something was wrong? Nope, this is straightforward.

Suppose $t = \sup(S)$ and $\varepsilon > 0$. Then $t - \varepsilon$ is not an upper bound for $S$. That is, there exists $s \in S$ such that $t - \varepsilon < s \leq t$. □