In this problem, we are asked to show that the Bessel function of order one given by
\[ J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \]
satisfies the differential equation
\[ x^2 J''_1(x) + x J'_1(x) + (x^2 - 1)J_1(x) = 0. \tag{1} \]

Our first order of business is to obtain series expressions for \( J'_1(x) \) and \( J''_1(x) \). By Wonderful Theorem we are free to differentiate \( J_1(x) \) once, and then a second time, without altering the radius of convergence. Doing so, we obtain
\[
J'_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)!2^{2n+1}} \quad \quad J''_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)x^{2n-1}}{n!(n+1)!2^{2n+1}}.
\]

Now, for the sake of combining these series expressions, we elect to reindex the expression for \( J''_1(x) \) so that the sum starts at \( n = 0 \) like the others:
\[
J''_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2(n+1)+1)(2(n+1))x^{2(n+1)-1}}{(n+1)!(n+2)!2^{2(n+1)+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+3)(2n+2)x^{2n+1}}{(n+1)!(n+2)!2^{2n+3}}.
\]

Substituting our expressions for \( J_1(x) \), \( J'_1(x) \) and \( J''_1(x) \) into the left side of equation (1), we obtain
\[
x^2 J''_1(x) + x J'_1(x) + (x^2 - 1)J_1(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+3)(2n+2)x^{2n+1}}{(n+1)!(n+2)!2^{2n+3}} + x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)!2^{2n+1}} + (x^2 - 1) \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+1}}{n!(n+1)!2^{2n+1}}
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+3)(2n+2)x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} + \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+3}}{n!(n+1)!2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+1}}{n!(n+1)!2^{2n+1}}.
\]

Note that two of these series are expressed in powers \( x^{2n+3} \) and two are
expressed in powers $x^{2n+1}$. Grouping these pairs together, we obtain

$$x^2 J''_1(x) + x J'_1(x) + (x^2 - 1) J_1(x) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1}(2n+3)(2n+2) + (-1)^{n}4(n+1)(n+2)}{(n+1)!(n+2)!2^{2n+3}} x^{2n+3} \right]
+ \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n+1) - (-1)^{n}n}{n!(n+1)!2^{2n+1}} x^{2n+1}
+ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1) - 2}{(n+1)!(n+2)!2^{2n+3}} x^{2n+3}
+ \sum_{n=0}^{\infty} \frac{(-1)^{n}[2n]x^{2n+1}}{n!(n+1)!2^{2n+1}}
= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} + \frac{(-1)^{n}[2n]x^{2n+1}}{n!(n+1)!2^{2n+1}} \right].$$

Finally, consolidating into a single sum we get the tidy expression

$$x^2 J''_1(x) + x J'_1(x) + (x^2 - 1) J_1(x) = \sum_{n=0}^{\infty} \left( \frac{(-1)^{n}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} + \frac{(-1)^{n}[2n]x^{2n+1}}{n!(n+1)!2^{2n+1}} \right).$$

The beauty of the above series is that it telescopes (write out the first few terms of the above series to see the start of the pattern)! To justify it generally, observe that the first term in the parentheses equals negative the second term in the next set of parentheses:

$$\frac{(-1)^{n+1}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} = -\frac{(-1)^{n}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}.$$

Thus, the typical partial sum of the series for $x^2 J''_1(x) + x J'_1(x) + (x^2 - 1) J_1(x)$ is

$$S_N = \sum_{n=0}^{N} \left( \frac{(-1)^{n}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} + \frac{(-1)^{n}[2n]x^{2n+1}}{n!(n+1)!2^{2n+1}} \right) = (-1)^{N}[2(N+1)]x^{2N+3}$$

which tends to zero as $N \to \infty$. Thus, we ultimately, magnificently, obtain

$$x^2 J''_1(x) + x J'_1(x) + (x^2 - 1) J_1(x) = \sum_{n=0}^{\infty} \left( \frac{(-1)^{n}[2(n+1)]x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} + \frac{(-1)^{n}[2n]x^{2n+1}}{n!(n+1)!2^{2n+1}} \right) = 0.$$