Sequential Limit Laws

Our precise \( \varepsilon \) definition of sequential convergence has many natural consequences. Chief among these are the limit laws for sequences, stated here for reference and provable with the precise definition (you will consider the proof of law (i) on Homework 1).

**Sequential Limit Laws** If \( \lim_{n \to \infty} a_n = L \), \( \lim_{n \to \infty} b_n = K \) and \( c \) is a constant, then

(i) \( \lim_{n \to \infty} (a_n + b_n) = L + K \)
(ii) \( \lim_{n \to \infty} (a_n - b_n) = L - K \)
(iii) \( \lim_{n \to \infty} c \cdot a_n = cL \)
(iv) \( \lim_{n \to \infty} a_n \cdot b_n = LK \)
(v) \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{K} \) (any caveats here on the \( b_n \) and/or \( K \)?)
(vi) \( \lim_{n \to \infty} a_n^p = L^p \) (any caveats here on the \( a_n \) and/or \( L \)?)

Today’s activity addresses potential downfalls of these limit laws, along with possible remedies.

**Exercise 1** Consider the sequence \( \{ne^{-n}\} \). Reasoning heuristically, what do you reckon is the limit?

**Exercise 2** Why doesn’t sequential limit law (iv) confirm your guess from exercise 1?

**Exercise 3** Let \( f(x) = xe^{-x} \). Use calculus to calculate \( \lim_{x \to \infty} f(x) \).

**Exercise 4** Why does your answer to exercise 3 definitively prove your answer to exercise 1?

The upshot of the first four exercises is that a product of two sequences may converge despite one or both of them failing to converge. Moreover, calculus may step to the fore to make quick work of a sequential limit problem. We need only think of the terms as certain values of a function; if the whole function has understandable “long run” behavior, so must the sequence that is nothing but a part of it.
Monotonic Sequences

Sometimes a sequence is not amenable to the sequential limit laws because it doesn’t have an easy formulaic description. Even in these cases, we can sometimes argue for convergence without even knowing the limit to which it converges. The key notion is that of monotonicity.

**Definition** The sequence \( \{a_n\} \) is called *increasing* if \( a_n \leq a_{n+1} \) for each \( n \geq 1 \). It is called *decreasing* if \( a_n \geq a_{n+1} \) for each \( n \geq 1 \).

You should compare and contrast this definition with that of Definition 10 in 12.1 of Stewart. We call his definitions *strictly increasing* and *strictly decreasing*. Most mathematicians would concur with our definition.

**Exercise 5** Classify each sequence as increasing, decreasing or neither. Justify your assertions.

(i) \( \left\{ \frac{n-1}{n} \right\} \)

(ii) \( \{ne^{-n}\} \)

(iii) \( \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\} \)

Certainly not all monotonic sequences converge (take \( \{n\} \), for example), but all *bounded* monotonic sequences converge.

**Definition** The sequence \( \{a_n\} \) is called *bounded above* if there is a number \( M \) such that \( a_n \leq M \) for each \( n \geq 1 \). It is called *bounded below* if there is a number \( m \) such that \( m \leq a_n \) for each \( n \geq 1 \). It is called *bounded* if it is bounded both above and below.

**Exercise 6** Prove (using the precise \( \varepsilon \) definition) that a bounded increasing sequence converges to the smallest number bigger than or equal to all of its terms.

**Exercise 7** Formulate and prove a statement akin to exercise 6 but for decreasing sequences.

The upshot of the last three exercises is that we can conclude that a sequence converges without knowing its limit in advance. We need only demonstrate monotonicity and boundedness and convergence must follow. This is the content of the Monotonic Sequence Theorem at the end of 12.1 in Stewart.

Equipped with our understanding of sequences, the remainder of chapter twelve is devoted to one deep and exceedingly useful kind of sequence: the infinite series.