$1,000 is put in the bank and earns 3% annual interest compounded continuously. In addition, during year 2, from $t = 1$ to $t = 2$, a total of $900$ dollars is added to the account by making steadily increasing daily deposits starting with $0$ in the first day. This yields the “deposit” function whose graph is shown below.

The “deposit function” $d(t) = \text{amount deposited at time } t$

How much money is in the bank after 5 years?

**Solution.** Denoting the money in the bank at time $t$ as $y(t)$,

$$y'(t) = 0.03y(t) + d(t), \quad y(0) = 1000.$$

Assuming the height of the triangle is $H$ its area is $1/2 \cdot H = 900$, so $H = 1800$. Consequently, the slope of the line is $1800$ and $d(t) = 1800(t - 1)$ provided $1 \leq t < 2$. In terms of the unit step function,

$$d(t) = 1800(t - 1)(u(t - 1) - u(t - 2)),$$

and the ivp is

$$y' = 0.03y + 1800(t - 1)(u(t - 1) - u(t - 2)), \quad y(0) = 1000.$$

Write the ode in the form $y' - 0.03y = 1800((t - 1)u(t - 1) - (t - 1)u(t - 2))$ and take the Laplace transform.

$$sY - 1000 - 0.03Y = 1800(e^{-s}/s^2 - e^{-2s}L\{1\})$$

$$= 1800\left(e^{-s}\cdot\frac{1}{s^2} - e^{-2s}\cdot\left(\frac{1}{s^2} + \frac{1}{s}\right)\right)$$

Solve for $Y$.

$$Y(s) = \frac{1000}{s - 0.03} + 1800\left(e^{-s}\cdot\left(\frac{1}{s^2(s - 0.03)}\right) - e^{-2s}\cdot\left(\frac{1}{s^2(s - 0.03)} + \frac{1}{s(s - 0.03)}\right)\right)$$

This inverts to

$$y(t) = 1000e^{0.03t} + 1800\left((t - 1)\cdot f(t - 1) - u(t - 2)(f(t - 2) + g(t - 2))\right)$$

where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s - 0.03)}\right\} = \int_0^t e^{0.03\tau}\,d\tau = \frac{1}{0.03}(e^{0.03t} - 1)$ and $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s - 0.03)}\right\} = \int_0^t g(\tau)\,d\tau = \int_0^t \frac{1}{0.03}(e^{0.03\tau} - 1)\,d\tau = \frac{1}{0.03}(e^{0.03t} - 1) - t/0.03$.

Substitute into the equation for $y(t)$ to obtain

$$y(t) = 1000e^{0.03t} + 1800\left(u(t - 1)\cdot\left(\frac{1}{0.0009}\left(e^{0.03(t - 1)} - 1\right) - (t - 1)/0.03\right) - u(t - 2)\cdot\left(\frac{1}{0.0009}\left(e^{0.03(t - 2)} - 1\right) - (t - 2)/0.03 + \frac{1}{0.03}\left(e^{0.03(t - 2)} - 1\right)\right)\right).$$

After 5 years there are $y(5) = 1000e^{0.15} + 2000000\left(e^{0.12} - e^{0.09}\right) - 600000e^{0.09} = 2156.51$ dollars in the bank. The graph of $y(t)$ is displayed below.