Math 241 (Camp) **Fall 2007**

Vector Identities

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$, for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at points where $g(\mathbf{x}) \neq 0$
5. $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$
6. $\nabla\times(\mathbf{F}+\mathbf{G}) = \nabla\times\mathbf{F} + \nabla\times\mathbf{G}$
7. $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$
8. $\nabla\cdot(f\mathbf{F}) = f\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla f$
9. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$
10. $\nabla\cdot(\nabla\times\mathbf{F}) = 0$
11. $\nabla\times(f\mathbf{F}) = f\nabla\times\mathbf{F} + \nabla f\times\mathbf{F}$
12. $\nabla\times(\mathbf{F}\times\mathbf{G}) = (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$
13. $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$
14. $\nabla\times\nabla f = \mathbf{0}$
15. $\nabla(\mathbf{F}\cdot\mathbf{F}) = 2(\mathbf{F}\cdot\nabla)\mathbf{F} + 2\mathbf{F}\times(\nabla\times\mathbf{F})$
16. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g)$
17. $\nabla\cdot(\nabla f\times\nabla g) = 0$
18. $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
19. $\mathbf{H}\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\mathbf{H}\times\mathbf{F}) = \mathbf{F}\cdot(\mathbf{G}\times\mathbf{H})$
20. $\mathbf{H}\cdot((\mathbf{F}\times\nabla)\times\mathbf{G}) = ((\mathbf{H}\cdot\nabla)\mathbf{G})\cdot\mathbf{F} - (\mathbf{H}\cdot\mathbf{F})(\nabla\cdot\mathbf{G})$
21. $\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = (\mathbf{F}\cdot\mathbf{H})\mathbf{G} - (\mathbf{F}\cdot\mathbf{G})\mathbf{H}$
22. $\nabla^2 f \equiv \nabla\cdot\nabla f$

Notes:

- f and g denote scalar fields; $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ denote vector fields.
- In Cartesian coordinates, the **vector Laplacian** is defined as $\nabla^2\mathbf{F} = \nabla^2 F_1\hat{\boldsymbol{\imath}} + \nabla^2 F_2\hat{\boldsymbol{\jmath}} + \nabla^2 F_3\hat{\boldsymbol{k}}$, where $\mathbf{F} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}} + F_3\hat{\boldsymbol{k}}$. To generalize to other coordinate systems, the vector Laplacian is often considered to be the defined by the relationship shown in identity #13 above.
- In Cartesian coordinates, the expression $\mathbf{V} = (\mathbf{F}\cdot\nabla)\mathbf{G}$ has, by definition, the components $V_i = \mathbf{F}\cdot(\nabla G_i)$ where $\mathbf{G} = G_1\hat{\boldsymbol{\imath}} + G_2\hat{\boldsymbol{\jmath}} + G_3\hat{\boldsymbol{k}}$. In other coordinates systems, the dependence of the unit vectors on position must be accounted for. (There's a good description of this in the 2nd appendix of *An Introduction to Fluid Dynamics* by G. K. Batchelor, Cambridge University Press, 1967.)
- In the expression $\mathbf{U} = (\mathbf{F}\times\nabla)\times\mathbf{G}$, the ∇ is to operate only on $\mathbf{G}$. In Cartesian coordinates, $\mathbf{U}$ has components $U_i = \mathbf{F}\times(\nabla G_i)$.