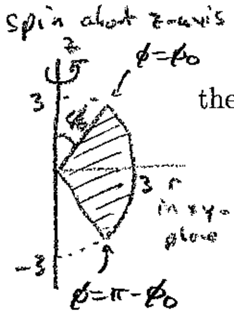
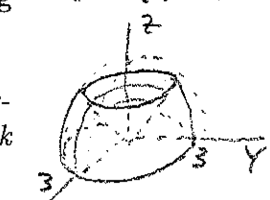
Math 241 (Camp) Fall ~~2007~~ 2010

A Partially Eaten Sphere
A triple integral in non-Cartesian coordinates



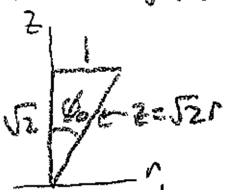


Let E be the solid formed by starting with the sphere $x^2+y^2+z^2=9$ and removing the solid bounded by the (double) cone $z^2=2\,(x^2+y^2)$.

1. Sketch E and find representations for it in both cylindrical and spherical coordinates. *Before actually determining the two representations, which do you think should be simpler?* Both the sphere and the cone have simpler representations in spherical than in cylindrical, so spherical will probably be simpler.

cylindrical	spherical
cone: $z^2=2r^2$	$\tan\phi=\frac{1}{2} \Rightarrow \phi\in\{\phi_0,\pi-\phi_0\}$ where $\phi_0=\arctan(\frac{1}{\sqrt2})$
sphere: $z^2=9-r^2$.	$\rho^2=9 \Rightarrow \rho=3$
intersection: $\begin{cases} 9-r^2=2r^2;\ z^2=2r^2 \\ \Rightarrow 3r^2=9;\ z=\pm2\sqrt3 \\ \Rightarrow r=\sqrt3,\ z=\pm2\sqrt3 \end{cases}$	$\rho=3,\ \phi=\phi_0$ above $z=0$ $\rho=3,\ \phi=\pi-\phi_0$ below $z=0$

* finding ϕ_0



$\Rightarrow \tan\phi_0=\frac{1}{\sqrt2}$

domain:

Cylindrical: $E:\{(r,\theta,z)\mid 0\leq r\leq 3,\ 0\leq\theta\leq 2\pi,\ \begin{cases}-\sqrt2 r\leq z\leq\sqrt2 r & \text{if } 0\leq r\leq\sqrt3\\ -\sqrt{9-r^2}\leq z\leq\sqrt{9-r^2} & \text{if } \sqrt3\leq r\leq 3\end{cases}$

Spherical: $E:\{(\rho,\theta,\phi)\mid 0\leq\rho\leq 3,\ 0\leq\theta\leq 2\pi,\ \phi_0\leq\phi\leq\pi-\phi_0\}$
when $\phi_0=\arctan\frac{1}{\sqrt2}$

2. Using one of the coordinate systems above, calculate the volume of E.

Clearly spherical is simpler (spherical wedge).

$$\text{volume } V(E)=\iiint_E dV=\int_0^{2\pi}\int_{\phi_0}^{\pi-\phi_0}\int_0^3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad \text{'separable'}$$

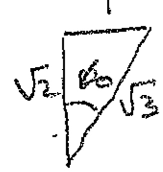
$$=\left(\int_0^{2\pi}d\theta\right)\left(\int_{\phi_0}^{\pi-\phi_0}\sin\phi\,d\phi\right)\left(\int_0^3\rho^2 d\rho\right)$$

$$=2\pi\left(-\cos\theta\Big|_{\phi_0}^{\pi-\phi_0}\right)\left(\tfrac13\rho^3\Big|_0^3\right)$$

$$=2\pi\left[\cos\phi_0-\cos(\pi-\phi_0)\right]\left(\tfrac{27}{3}\right)$$

$$=18\pi\left[2\cos\phi_0\right]$$

$$\boxed{=36\pi\sqrt{\tfrac23}}$$



$\tan\phi_0=\frac{1}{\sqrt2} \Rightarrow \cos\phi_0=\sqrt{\frac23}$

$(\sin\phi_0=\frac{1}{\sqrt3})$

note: $\cos(\pi-\phi)=-\cos\phi$

3. Using the same coordinate system, setup and calculate the triple integral

$$I = \iiint_E y^2 z^2 \, \mathrm{d}V$$

use spherical again, need integrand in spherical

$$f(x,y,z) = y^2 z^2 = (\rho \sin\phi \sin\theta)^2 (\rho\cos\phi)^2 = \rho^4 \sin^2\phi \cos^2\phi \sin^2\theta$$

$$\Rightarrow I = \int_0^{2\pi} \int_{\phi_0}^{\pi-\phi_0} \int_0^3 \rho^6 \sin^3\phi \cos^2\phi \sin^2\theta \, d\rho \, d\phi \, d\theta \qquad \text{'seperable'}$$

$$= \left[\int_0^{2\pi} \sin^2\theta \, d\theta\right] \left[\int_{\phi_0}^{\pi-\phi_0} \sin^3\phi \cos^2\phi \, d\phi\right] \left[\int_0^3 \rho^6 \, d\rho\right]$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} [1-\cos 2\theta] d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]\Big|_0^{2\pi} = \frac{1}{2}(2\pi) = \pi$$

($\sin 4\pi = \sin 0 \Rightarrow$ cancel)

$\int \sin^3\phi \cos^2\phi \, d\phi$; let $u = \cos\phi$, $\sin^2\phi = 1-\cos^2\phi = 1-u^2$, $du = -\sin\phi \, d\phi$

$$= \int -(1-u^2)u^2 du = \int (u^4 - u^2) du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{\cos^5\phi}{5} - \frac{\cos^3\phi}{3} + C$$

$$\therefore \int_{\phi_0}^{\pi-\phi_0} \sin^3\phi \cos^2\phi \, d\phi = \frac{1}{5}\left[\cos^5(\pi-\phi_0) - \cos^5(\phi_0)\right] - \frac{1}{3}\left[\cos^3(\pi-\phi_0) - \cos^3(\phi_0)\right]$$

$$= -\frac{2}{5}\cos^5(\phi_0) + \frac{2}{3}\cos^3(\phi_0) = \frac{2}{3}\left(\frac{2}{3}\right)^{3/2} - \frac{2}{5}\left(\frac{2}{3}\right)^{5/2}$$

$$= \left(\frac{2}{3}\right)^{5/2}\left[1 - \frac{2}{5}\right) = \frac{4\sqrt{2}}{9\sqrt{3}} \cdot \frac{3}{5} = \frac{4\sqrt{6}}{45}$$

$$\int_0^3 \rho^6 d\rho = \frac{1}{7}\rho^7\Big|_0^3 = \frac{3^7}{7}$$

$$\therefore I = \pi \cdot \frac{4\sqrt{6}}{\cancel{45}_{5}} \cdot \frac{\cancel{3^7}\,3^5}{7} = \frac{\pi\, 4\sqrt{6} \cdot 243}{35} = \frac{972\sqrt{6}\,\pi}{35}$$