Math 241 (Camp)

Triple Integral Worksheet

Let

$$I = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$$

1. Define E algebraically and then find and sketch the 6 potential bounding surfaces for the region of integration E. *Generally, all 6 surfaces will not actually appear as sides of the solid; this is particular the case for the surfaces defined by the outer variable.*

Directly drawn from the given bounds of integration, we get

$$E: \{(x,y,z) \mid \underbrace{0 \le x \le 1}_{\text{"outer"}},\ \underbrace{\sqrt{x} \le y \le 1}_{\text{"middle"}},\ \underbrace{0 \le z \le 1-y}_{\text{"inner"}}\}$$

The potential bounding surfaces are

$z = 0$ and $z = 1-y$ — Since these are the inner variable, these <u>will</u> appear as boundary surfaces for E.

$\uparrow$ horizontal plane, $\uparrow$ tilted plane

$y = \sqrt{x}$ curved vertical surface

$y = 1$

$x = 0$

$y = 1$

$\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right\}$ vertical planes

these <u>might</u> appear as boundary surfaces for E, but they might not.

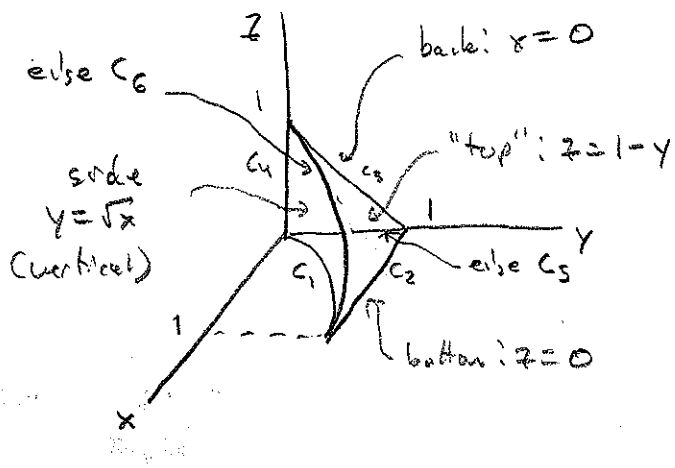
<u>No</u> other surfaces can be boundary surfaces ("sides") of a solid occupying E.

2. Sketch E. *If you find this difficult, start the next step: finding the 3 2D-projections. Often, it takes going back-and-forth between the 3D-solid and the 2D-projections to get the full picture.*

Given order has z as inner variable $\Rightarrow$ try to find projection D in x-y first

$E: \{(x,y,z) \mid (x,y) \in D_1,\ 0 \le z \le 1-y\}$ where $D_1: \{(x,y,z) \mid 0 \le x \le 1, \sqrt{x} \le y \le 1\}$

[see #3 for sketch of D_1 in x-y plane]



4 bounding sides: $z=0$, $x=0$, $y=\sqrt{x}$ and $z=1-y$

6 edges: defined by intersecting sides

C_1: $y=\sqrt{x}$ and $z=0$

C_2: $z=1-y$ and $z=0$

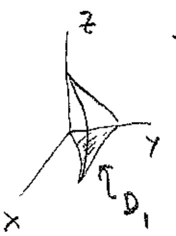
C_3: $z=1-y$ and $x=0$

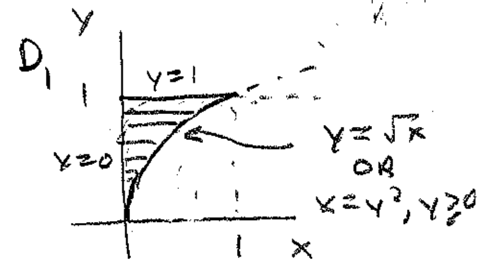
C_4: $x=0$ and $y=\sqrt{x}$

C_5: $x=0$ and $z=0$

C_6: $y=\sqrt{x}$ and $z=1-y$

3. Find and sketch the 3 projections of E onto the coordinate planes: x-y, x-z, y-z.
Find and label all curves for each 2D region.
What are the bounds for the inner variable for each of these projections?
Remember, the edge curves of the solid are the intersections of two of the bounding surfaces of the solid. You might find it easiest to start with the projection in z onto the x-y plane, since that's the order of the given integral.





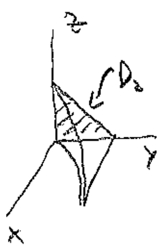
<u>z-inner</u>: $E: \{(x,y,z) \mid (x,y)\in D_1,\ 0\leq z\leq 1-y\}$

where $D_1: \{(x,y) \mid 0\leq x\leq 1,\ \sqrt{x}\leq y\leq 1\}$

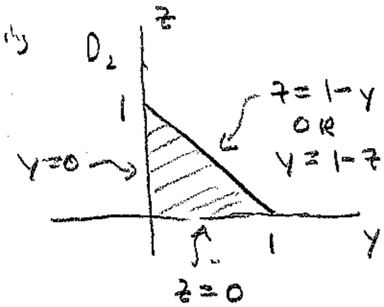
This is a "y-simple" definition of D_1.
Alternatively, we could define D_1 as "x-simple"

$\Rightarrow D_1: \{(x,y) \mid 0\leq y\leq 1,\ 0\leq x\leq y^2\}$

To get the other projections, we need to look at the solid E (see #2).



<u>x-inner</u>: If we want to define E w/ x as the inner variable, we're projecting E onto the y-z plane. For this solid, that's actually a side of the solid. The boundary curves for D_2 come from the edges of E.



$D_2: \{(y,z) \mid 0\leq y\leq 1,\ 0\leq z\leq 1-y\}$

OR

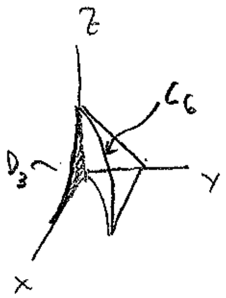
$D_2: \{(y,z) \mid 0\leq z\leq 1,\ 0\leq y\leq 1-z\}$

For any $(y,z)\in D_2$, consider a line parallel to the x-axis through that point. As x increases on that line, we enter the solid through the side $x=0$ and leave through $y=\sqrt{x}$ (or $x=y^2$)

$\Rightarrow 0\leq x\leq y^2$

<u>y-inner</u>: Projecting onto the x-z plane, we need the projection of the edge C_6

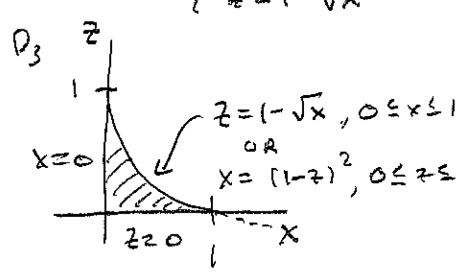
$C_6: \begin{cases} y=\sqrt{x} \\ z=1-y \end{cases} \Rightarrow \begin{cases} y=\sqrt{x} \\ z=1-\sqrt{x} \end{cases}$; so the projection in y is $\begin{cases} y=0 \\ z=1-\sqrt{x} \end{cases}$



$\Rightarrow D_3: \{(x,z) \mid 0\leq x\leq 1,\ 0\leq z\leq 1-\sqrt{x}\}$

OR

$D_3: \{(x,z) \mid 0\leq z\leq 1,\ 0\leq x\leq (1-z)^2\}$



and the bounds in y are

$\sqrt{x}\leq y\leq 1-z$

[Consider a line parallel to y axis through any point in D_3 as y increases]

4. Rewrite in integral as an equivalent iterated integral in the 5 other orders of integration. *Again, you might find it easiest to start with the projection in z onto the x-y plane but reverse the order if integration between x and y, i.e., $dV = dz\,dx\,dy$. Then do both choices for each of the other projections.*

All the algebraic work was done in #3.

Given $\underline{dV = dz\,dy\,dx}$: $\Rightarrow$ $I = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$

Swapping y, x but leaving z as inner uses same projection D_1 but redefined

$\underline{dV = dz\,dx\,dy}$: $I = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$

Note: bounds for z unchanged.

Projecting onto D_2, x-inner

$\underline{dV = dx\,dz\,dy}$: $I = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$

$\underline{dV = dx\,dy\,dz}$: $I = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$

Again, inner variable's bound are the same for these two.

Projecting onto D_3, y-inner

$\underline{dV = dy\,dz\,dx}$: $I = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$

$\underline{dV = dy\,dx\,dz}$: $I = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$

And again, inner bounds are unchanged.