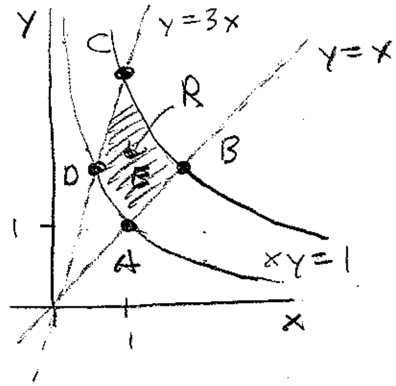
a) let $I = \iint_R xy \, dA$; R is domain in 1st quadrant bounded by

lines $y=x$, $y=3x$; hyperbolae $xy=1$, $xy=3$



intersection points:

A: $\begin{cases} y=x \\ xy=1 \end{cases} \Rightarrow x^2=1 \Rightarrow x=1$ (1st Q) $\Rightarrow (1,1)$

B: $\begin{cases} y=x \\ xy=3 \end{cases} \Rightarrow x^2=3 \Rightarrow x=\sqrt{3} \Rightarrow (\sqrt{3},\sqrt{3})$

C: $\begin{cases} y=3x \\ xy=3 \end{cases} \Rightarrow 3x^2=3 \Rightarrow \begin{cases} x=1 \\ y=3 \end{cases}$ D: $\begin{cases} y=3x \\ xy=1 \end{cases} \Rightarrow 3x=1 \Rightarrow \begin{cases} x=\frac{1}{\sqrt{3}} \\ y=\sqrt{3} \end{cases}$

b) evaluate integral using the transformation

$$(x,y) = T(u,v) : \begin{cases} x = u/v \\ y = v \end{cases}$$

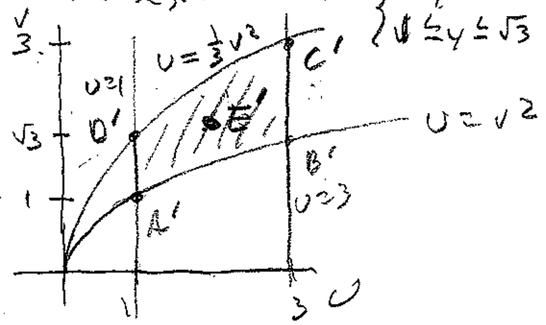
i) find S, <u>domain</u> in u-v plane, image of (R)

line segment AB: $\begin{cases} y=x \\ 1 \le y \le \sqrt{3} \end{cases} \Rightarrow \begin{cases} v = \frac{u}{v} \\ 1 \le v \le \sqrt{3} \end{cases} \Rightarrow \begin{cases} u = v^2 \\ 1 \le v \le \sqrt{3} \end{cases}$

line segment BC: $\begin{cases} xy=3 \\ \sqrt{3} \le y \le 3 \end{cases} \Rightarrow \begin{cases} u=3 \\ \sqrt{3} \le v \le 3 \end{cases}$

line segment CD: $\begin{cases} y=3x \\ \sqrt{3} \le y \le 3 \end{cases} \Rightarrow \begin{cases} v = 3\frac{u}{v} \\ \sqrt{3} \le v \le 3 \end{cases} \Rightarrow \begin{cases} u = \frac{1}{3}v^2 \\ \sqrt{3} \le v \le 3 \end{cases}$

line segment DA: $\begin{cases} xy=1 \\ 1 \le y \le \sqrt{3} \end{cases} \Rightarrow \begin{cases} u=1 \\ 1 \le v \le \sqrt{3} \end{cases}$



check $(u,v) = (2,2)$

maps to $T(u,v) = (\frac{2}{2}, 2) = (1,2)$

$E(1,2) \Longleftrightarrow E'(2,2)$

we've mapped interior to interior

ii) transform integrand

$$f(x,y) = xy \Rightarrow f(x(u,v), y(u,v)) = \frac{u}{v} \cdot v = u$$

iv) transform $dA \Rightarrow$ Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} - (0) = \frac{1}{v}$$

note $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ in S (and bounded)

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \left| \frac{1}{v} \right| du\,dv = \frac{1}{v}\,du\,dv$$

(v) $\therefore$ $I = \iint\limits_R xy\,dA = \iint\limits_S u \cdot \frac{1}{v}\,du\,dv$

S ← order undetermined here

given our S, easiest definition is $\begin{cases} 1 \leq u \leq 3 & \text{outer} \\ \sqrt{u} \leq v \leq \sqrt{3u} & \text{inner} \end{cases}$

$$\Rightarrow I = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dv\,du = \int_1^3 u \ln v \Big|_{v=\sqrt{u}}^{\sqrt{3u}} du$$

$$= \int_1^3 u \left[\ln\sqrt{3u} - \ln\sqrt{u} \right] du = \int_1^3 u \ln\left[\frac{\sqrt{3u}}{\sqrt{u}} \right] du$$

$$= \int_1^3 u \ln\sqrt{3}\,du = \frac{1}{2}\ln\sqrt{3} \cdot u^2 \Big|_{u=1}^{3} = \frac{1}{4}\ln 3 \cdot (9-1)$$

$$= \boxed{2\ln 3}$$

note: we never directly used it here, but T is invertible on this domain

$$\begin{cases} x = \frac{u}{v} \\ y = v \end{cases} \Rightarrow \begin{cases} u = xv \\ v = y \end{cases} \Rightarrow \boxed{\begin{cases} u = xy \\ v = y \end{cases} \equiv T^{-1}(x,y)}$$