Partial Differentiation and Classifying Critical Points of Surfaces

- **Partial Derivatives**

  Given a function of 2 variables, $z = f(x, y)$, the partial derivatives are found by differentiating with respect to one variable while holding the other constant; since we have two input variables, there are two first partial derivatives. The limit definitions are

$$f_x = \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_y = \frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

  In practice, we don't use the limit definition, we use standard formulas for differentiation - just consider the other variable to be a fixed parameter.

  The second partial derivatives are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$
$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

  Note that the order for the "mixed" partials is generally not important: $f_{xy} = f_{yx}$ wherever both f_{xy} and f_{yx} are continuous (Clairaut's Theorem).

- **Finding and Classifying Critical Points**

  At the relative extrema, both maxima and minima, both of these first partials will be zero if they exist. The tangent plane to the surface at a relative max or min is flat, so all tangents, which lie in the tangent plane, have zero slope. Turning this around, we have a test for POTENTIAL extrema; *i.e.*, where both of the first partials are zero. To catch all potential extrema, we must also consider points at which one or both first partial derivatives does not exist. This is the definition of a critical point:

  A point (a, b) in the domain of $f(x, y)$ is a **critical point** if either

  - $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or
  - at least one of first partial derivatives does not exist at (a, b).

  But just being a critical point does not guarantee that the point is a local extrema. It might also be a saddle point (sort of a min in one direction, max in another) or an even more complicated shape. So we need to find a test which attempts to classify the critical point as a max, a min or as a saddle. Such a test exists if the first partial derivatives exist at the critical point – it uses the second partial derivatives:

The Second Derivative Test (for a 2 variable function)
Let the point $(x, y) = (a, b)$ be a critical point of $z = f(x, y)$ where

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

and where the second partial derivatives are continuous about $(a,\ b)$.
Then, if we evaluate the following expression at the critical point, (a, b), we can determine the type of critical point it is:

$$D = f_{xx} f_{yy} - [f_{xy}]^2 \Big|_{(x,y)=(a,b)}$$

The sign of D (a number) determines the type of critical point:

$$\text{if } \begin{cases} D > 0 & \text{then } (a, b) \text{ is a relative extrema;} \\ D < 0 & \text{then } (a, b) \text{ is a saddle;} \\ D = 0 & \text{then the test is inconclusive.} \end{cases}$$

If $D > 0$, we have a follow-up test to determine if the extrema is a max or a min:

$$\text{if } D > 0 \text{ and } \begin{cases} f_{xx}(a, b) < 0 & \text{then } (a, b) \text{ is a relative max;} \\ f_{xx}(a, b) > 0 & \text{then } (a, b) \text{ is a relative min.} \end{cases}$$

A couple of examples:

1. Find the location of all relative extrema and saddle points for

$$f(x, y) = 4xy + 8x - 9y$$

   (a) Identify the critical points (potential extrema) by setting both first partials to zero and solving the set of resulting equations. The 1st partials are

$$f_x(x, y) = 4y + 8, \qquad f_y(x, y) = 4x - 9;$$

   setting these equal to zero we get a system of equations which is easy to solve:

$$\begin{cases} 4y + 8 = 0 \\ 4x - 9 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} y = -2 \\ x = \frac{9}{4} \end{cases}$$

   so we have a single critical point located at $(x, y) = (9/4, -2)$

(b) Classify each critical points be evaluating D using the 2nd partial derivatives evaluated at the critical point. The 2nd partials are:

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x(x,y)) = 0,$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y(x,y)) = 0,$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x(x,y)) = 4.$$

So, at the critical point $(9/4,-2)$, we get

$D = f_{xx}(9/4,-2)\cdot f_{yy}(9/4,-2) - [f_{xy}(9/4,-2)]^2 = 0\cdot 0 - (4)^2 = -16 < 0.$

Therefore, $f(x,y) = 4xy + 8x - 9y$ has a saddle point at $(x,y) = (9/4,-2)$ and no local extrema.

This is just the location of the saddle point in the "input" x-y plane, the coordinates of the saddle point on the surface in 3D are

$$(x,y,z) = (a,b,f(a,b)) = (9/4,-2,f(9/4,-2)) = (9/4,-2,18).$$

2. Find the location of all relative extrema and saddle points for

$$f(x,y) = 5x^3 + 2y^2 - 60xy - 3$$

(a) Critical points:

$$f_x(x,y) = 15x^2 - 60y \qquad f_y(x,y) = 4y - 60x$$

Setting to zero:

$$\begin{cases} 15x^2 - 60y = 0 \\ 4y - 60x = 0 \end{cases} \implies \begin{cases} x^2 - 4y = 0 \\ y - 15x = 0 \end{cases}$$

So, we have $y = 15x$ which we can substitute into $x^2 - 4y = 0$, to get

$x^2 - 4(15x) = 0 \implies x^2 - 60x = 0 \implies x(x-60) = 0 \implies x \in 0, 60$

Using the substitution expression, $y = -15x$, we get 2 critical points at $(0,0)$ and $(60,900)$.

(b) Classify each critical point:

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(15x^2 - 60y\right) = 30x$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}(4y - 60x) = 4$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(15x^2 - 60y\right) = -60$$

Evaluating at <u>each</u> critical point, we get

- $(x, y) = (0, 0)$:

  $$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 \cdot 4 - (-60)^2 = -60^2 = -3600 < 0;$$

  therefore $(0, 0)$ is a saddle point.
- $(x, y) = (60, 900)$:

  $$D = f_{xx}(60,900) \cdot f_{yy}(60,900) - [f_{xy}(60,900)]^2 = 30 \cdot 60 \cdot 4 - (-60)^2 = 3,600 > 0.$$

  So $(x, y) = (60, 900)$ is a relative extremum; checking $f_{xx}(60, 900) = 20 > 0$, further classifies it as a local minimum.

So $f(x, y) = 5x^3 + 2y^2 - 60xy - 3$ has a local minimum located at $(60, 900)$ and a saddle point located at $(0, 0)$.