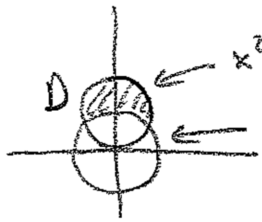
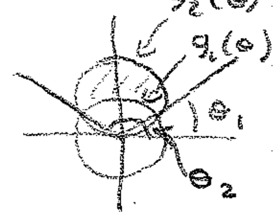
5.1.6.5 p1054

14) lamina inside $x^2+y^2=2y$ and outside $x^2+y^2=1$ w/ $\rho = \frac{k}{\sqrt{x^2+y^2}}$

[note: I misread problem in class as <u>inside</u> both circles, a harder problem]



D here is neither a type I nor type II in cartesian, so lets try polar



$x^2+y^2=1 \Rightarrow r^2=1$ or $\boxed{r=1}$

$x^2+y^2=2y \Rightarrow r^2=2r\sin\theta \Rightarrow \boxed{r=2\sin\theta}$ for $0\le\theta\le\pi$

this is a good type II polar region

$D: \{(r,\theta) \mid \theta_1\le\theta\le\theta_2,\ g_1(\theta)\le r\le g_2(\theta)\}$

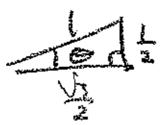
we need both $g_1(\theta)$ & $g_2(\theta)$ but we've already got them

$g_1(\theta)=1$ since $x^2+y^2=1 \Rightarrow r=1$

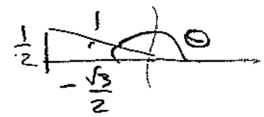
$g_2(\theta)=2\sin\theta$ " $x^2+y^2=2y \Rightarrow r=2\sin\theta$

still need to find θ_1 & θ_2

just set $g_1(\theta)=g_2(\theta) \Rightarrow 2\sin\theta=1 \Rightarrow \sin\theta=\frac{1}{2}$



i.e. $\theta=\frac{\pi}{6}$ or $\theta=\frac{5\pi}{6}$



So $D:\ \frac{\pi}{6}<\theta<\frac{5\pi}{6},\ 1<r<2\sin\theta$

Now for density in polar, we get $\rho=\frac{k}{r}$, note we're far from the origin in D, so $r\neq 0$.

So the mass integral becomes

$$\iint_D \rho(x,y)\,dA = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta}\left(\frac{k}{r}\right)\underbrace{r\,dr\,d\theta}_{dA \text{ via polar}}$$

$$m = k\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} dr\,d\theta = k\int_{\pi/6}^{5\pi/6}(2\sin\theta - 1)\,d\theta$$

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos\theta\Big|_{\pi/6}^{5\pi/6} = -\sqrt{3}$

$$= k\left[-2\cos\theta - \theta\Big|_{\pi/6}^{5\pi/6}\right] = k\left[2\sqrt{3} - \frac{2\pi}{3}\right]$$

note: $m \simeq 1.370k > 0$ mass is positive ✓

from symmetry: $M_y = \iint_D x\,\rho(x,y)\,dV = 0$

since lamina is symmetric in both shape & density about $y=0$

$\left[\text{can confirm, evaluate } \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} r\cos\theta\left(\frac{k}{r}\right) r\,dr\,d\theta \text{. should be } 0\right]$

$$M_x = \iint_D y\,\rho(x,y)\,dV = \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} r\sin\theta\left(\frac{k}{r}\right) r\,dr\,d\theta$$

$$= k\int_{\pi/6}^{5\pi/6}\sin\theta\int_{1}^{2\sin\theta} r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6}\sin\theta\left(\frac{1}{2}r^2\Big|_{r=1}^{2\sin\theta}\right)d\theta$$

$$= \frac{k}{2}\int_{\pi/6}^{5\pi/6} 4\sin^3\theta - \sin\theta\,d\theta$$

aside $\int \sin^3\theta\,d\theta = \int(1-\cos^2\theta)\sin\theta\,d\theta$

$= \int\sin\theta\,d\theta - \int\underbrace{\cos^2\theta}_{u^3}\underbrace{\sin\theta\,d\theta}_{-du}$

$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

$$= \frac{k}{2}\left[\frac{4}{3}\cos^3\theta - 4\cos\theta + \cos\theta\right]_{\theta=\pi/6}^{5\pi/6}$$

$$= \frac{k}{2}\left[\frac{4}{3}\cos^3\theta - 3\cos\theta\right]_{\theta=\pi/6}^{5\pi/6}$$

$\cos\pi/6 = \frac{\sqrt{3}}{2}$, $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

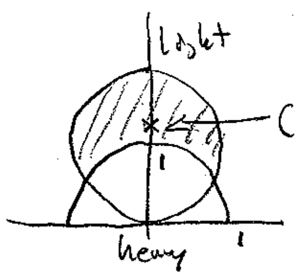
$\cos\theta\Big|_{\pi/6}^{5\pi/6} = -\sqrt{3}$

$\cos^3\theta\Big|_{\pi/6}^{5\pi/6} = -\frac{\sqrt{27}}{8} - \frac{\sqrt{27}}{8} = -\frac{3\sqrt{3}}{4}$

$$= \frac{k}{2}\left[\frac{4}{3}\left(-\frac{3}{4}\sqrt{3}\right) - 3\left(-\sqrt{3}\right)\right]$$

$$= \sqrt{3}\,k$$

$$\therefore\ \bar{x} = \frac{M_y}{m} = \boxed{0},\quad \bar{y} = \frac{M_x}{m} = \frac{\sqrt{3}\,k}{k\left(2\sqrt{3} - \frac{2\pi}{3}\right)} = \boxed{\frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)}} \simeq 1.26$$



$(\bar{x}, \bar{y})$ which seems quite reasonable.