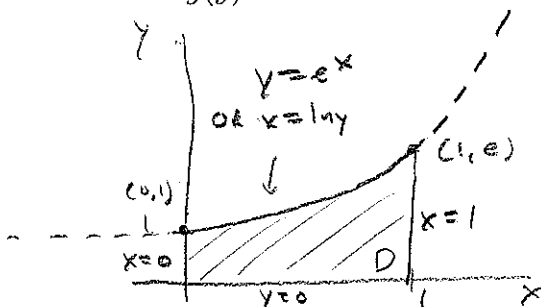


Given a lamina of density  $\rho(x, y) = y$  on the region bounded by  $y = e^x$ ,  $y = 0$ ,  $x = 0$  and  $x = 1$ , do the following:

1. Sketch the region occupied by the lamina. Label all boundary curves by their given algebraic definition. Also label the  $y = e^x$  curve with its alternate definition in the form  $x = g(y)$ .



2. Algebraically define the region occupied by the lamina using both orders, i.e., both as a  $x$ -simple region and as a  $y$ -simple region.

$y$ -simple:  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq e^x\}$

$x$ -simple:  $D = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq 1 \text{ OR } 1 \leq y \leq e, \ln y \leq x \leq 1\}$

note: the second definition is piecewise since the "left" boundary  $e^{\frac{1}{}}$  is piecewise defined as  $\begin{cases} x = \ln y & \text{if } 1 \leq y \leq e \\ x = 0 & \text{if } 0 \leq y \leq 1 \end{cases} \Rightarrow$

3. Using either definition of the domain  $D$ , find the mass of the lamina.

The " $y$ -simple" definition is clearly simpler to work with here

$$\begin{aligned}
 m &= \iint_D \rho(x, y) \, dA = \int_0^1 \int_0^{e^x} y \, dy \, dx \\
 &= \int_0^1 \left[ \frac{1}{2} y^2 \right]_{y=0}^{e^x} dx = \frac{1}{2} \int_0^1 e^{2x} \, dx \\
 &= \frac{1}{4} e^{2x} \Big|_{x=0}^1 = \frac{1}{4} (e^2 - 1) \approx 1.60
 \end{aligned}$$

(exact)                      (approx.)

↑  
Use this exact value, not the approximate value, in later calculations. Avoids propagation and potential amplification of error.

More on next page.

4. Find the first moments and the center of mass of the lamina.

$$M_y = \iint_D x \rho \, dA = \int_0^1 \int_0^{e^x} x y \, dy \, dx = \int_0^1 \frac{1}{2} x y^2 \Big|_{y=0}^{e^x} dx$$

$$= \frac{1}{2} \int_0^1 x e^{2x} dx \quad (\text{I.B.P. } u = x \quad du = e^{2x} dx)$$

$$= \frac{1}{4} x e^{2x} - \frac{1}{8} e^{2x} \Big|_{x=0}^1 = \left(\frac{1}{4} - \frac{1}{8}\right) e^2 - \left(0 - \frac{1}{8}\right) = \frac{e^2 + 1}{8}$$

$$M_x = \iint_D y \rho \, dA = \int_0^1 \int_0^{e^x} y^2 \, dy \, dx = \frac{1}{3} \int_0^1 y^3 \Big|_{y=0}^{e^x} dx = \frac{1}{3} \int_0^1 e^{3x} dx$$

$$= \frac{1}{9} e^{3x} \Big|_{x=0}^1 = \frac{e^3 - 1}{9}$$

$$\bar{x} = \frac{1}{m} M_y = \frac{4}{e^2 - 1} \cdot \frac{e^2 + 1}{8} = \frac{e^2 + 1}{2(e^2 - 1)} \approx 0.66$$

$$\bar{y} = \frac{1}{m} M_x = \frac{4}{e^2 - 1} \cdot \frac{e^3 - 1}{9} = \frac{4}{9} \left( \frac{e^3 - 1}{e^2 - 1} \right) = \frac{4}{9} \left( \frac{e^2 + e + 1}{e + 1} \right) \approx 1.33$$

5. Find the moments of inertia and the radii of gyration of the lamina about the  $x$ -axis,  $y$ -axis and the origin.

$$I_x = \iint_D y^2 \rho \, dA = \int_0^1 \int_0^{e^x} y^3 \, dy \, dx = \frac{1}{4} \int_0^1 y^4 \Big|_{y=0}^{e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx$$

$$= \frac{1}{16} e^{4x} \Big|_{x=0}^1 = \frac{e^4 - 1}{16}$$

$$I_y = \iint_D x^2 \rho \, dA = \int_0^1 \int_0^{e^x} x^2 y \, dy \, dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx$$

I.B.P.  $u = x^2 \quad du = 2x dx$   
 $du = 2x dx \quad v = \frac{1}{2} e^{2x}$

$$= \frac{1}{4} x^2 e^{2x} \Big|_0^1 - \underbrace{\frac{1}{2} \int_0^1 x e^{2x} dx}_{\text{use result in part (4)}} = \frac{1}{4} e^2 - \left(\frac{e^2 + 1}{8}\right) = \frac{e^2 - 1}{8}$$

$$I_0 = I_x + I_y = \frac{e^4 - 1}{16} + \frac{e^2 - 1}{8} = \frac{e^2 - 1}{16} (e^2 + 1 + 2) = \frac{(e^2 - 1)(e^2 + 3)}{16}$$

$$(\bar{\bar{x}})^2 = \frac{I_y}{m} = \frac{4}{e^2 - 1} \cdot \frac{e^2 - 1}{8} = \frac{1}{2} \Rightarrow \bar{\bar{x}} = \sqrt{\frac{1}{2}} \approx 0.707$$

$$(\bar{\bar{y}})^2 = \frac{I_x}{m} = \frac{4}{e^2 - 1} \cdot \frac{e^4 - 1}{16} = \frac{1}{4} (e^2 + 1) \Rightarrow \bar{\bar{y}} = \frac{1}{2} \sqrt{e^2 + 1} \approx 1.45$$

$$(\bar{\bar{r}})^2 = \frac{I_0}{m} \Rightarrow (\bar{\bar{r}})^2 = (\bar{\bar{x}})^2 + (\bar{\bar{y}})^2 \quad \text{ie} \quad \bar{\bar{r}} = \|\langle \bar{\bar{x}}, \bar{\bar{y}} \rangle\| = \frac{1}{2} \sqrt{e^2 + 3} \approx 1.61$$